Proof Formats for CS350

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In this note I recommend a format that has a minimal amount of English wrapped around the mathematics — enough so that we can follow the argument, but not so much that it's tedious to write.

Recommended format

We are required to prove that $\Theta(g(n)) = O(g(n)) \cap \Omega(g(n))$. The proof is in two parts.
First we prove that $\Theta(g(n)) \subseteq O(g(n)) \cap \Omega(g(n))$:

$$f(n) \in \Theta(g(n)) \quad \text{[assumption]} \tag{1}$$
$$\exists c_1, c_2, n_0 : \forall n > n_0 : c_1 g(n) \leq f(n) \leq c_2 g(n) \quad \text{[def. of } \Theta\text{]} \tag{2}$$
$$f(n) \in O(g(n)) \quad \text{[def. of } O \text{ and (2), letting } c = c_2\text{]} \tag{3}$$
$$f(n) \in \Omega(g(n)) \quad \text{[def. of } \Omega \text{ and (2), letting } c = c_1\text{]} \tag{4}$$
$$\Theta(g(n)) \subseteq O(g(n)) \cap \Omega(g(n)) \quad [\forall f : f \in \Theta \Rightarrow f \in O \wedge f \in \Omega] \tag{5}$$

Then we prove that $\Theta(g(n)) \supseteq O(g(n)) \cap \Omega(g(n))$:

$$f(n) \in O(g(n)) \quad \text{[assumption]} \tag{6}$$
$$f(n) \in \Omega(g(n)) \quad \text{[assumption]} \tag{7}$$
$$\exists c_1, n_1 : \forall n > n_1 : f(n) \geq c_1\, g(n) \quad \text{[def. of } \Omega\text{]} \tag{8}$$
$$\exists c_2, n_0 : \forall n > n_0 : f(n) \leq c_2\, g(n) \quad \text{[def. of } O\text{]} \tag{9}$$
$$\exists c_1, c_2, n_0 : \forall n > \max(n_0, n_1) : c_1\, g(n) \leq f(n) \leq c_2\, g(n) \quad \text{[combining (8) and (9) above]} \tag{10}$$
$$f(n) \in \Theta(g(n)) \quad \text{[def. of } \Theta \text{ and (10)]} \tag{11}$$
$$\Theta(g(n)) \supseteq O(g(n)) \cap \Omega(g(n)) \quad [\forall f : f \in O \wedge f \in \Omega \Rightarrow f \in \Theta] \tag{12}$$

Combining these subset (5) and superset (12) results, we get

$$\Theta(g(n)) = O(g(n)) \cap \Omega(g(n)) \tag{13}$$

because $A \subseteq B \wedge A \supseteq B \Rightarrow A = B$

Q.E.D.

Alternative format with more English

This alternative includes all of the math above, plus a lot more English explanation. I don't have a problem with you putting in *more* explanation than given above. Just remember, that the explanation doesn't *replace* the mathematics — the explanation *supplements* the mathematics.

We are required to prove that $\Theta(g(n)) = O(g(n)) \cap \Omega(g(n))$. The proof is in two parts; here I'm showing just the first part, because it's wordy.

First we prove that $\Theta(g(n)) \subseteq O(g(n)) \cap \Omega(g(n))$:

- We start by considering an arbitrary function $f(n)$ that is in $\Theta(g(n))$

$$f(n) \in \Theta(g(n)) \tag{14}$$

- From the definition of Θ we know that this means:

$$\exists c_1, c_2, n_0 : \forall n > n_0 : c_1\, g(n) \leq f(n) \leq c_2\, g(n) \tag{15}$$

- If we elide the first inequality, and substitute c for c_2, this gives us

$$\exists c, n_0 : \forall n > n_0 : f(n) \leq c\, g(n) \tag{16}$$

  which, from the definition of O, tells us

$$f(n) \in O(g(n)) \tag{17}$$

- Similarly, if we elide the second inequality, and substitute c for c_1, this gives us

$$\exists c, n_0 : \forall n > n_0 : c\, g(n) \leq f(n) \tag{18}$$

  which, from the definition of Ω, tells us

$$f(n) \in \Omega(g(n)) \tag{19}$$

- From the definition of $\cap$ for sets, we know that line 17 and line 19 together imply that

$$f(n) \in O(g(n)) \cap \Omega(g(n)) \tag{20}$$

- Recall the assumption (line 14) that $f(n)$ was an arbitrary function in $\Theta(g(n))$. So we have, using line 14, the rule of universal generalization, and line 20

$$\forall f : f(n) \in \Theta(g(n)) \implies f(n) \in O(g(n)) \cap \Omega(g(n)) \tag{21}$$

  This implies that

$$\Theta(g(n)) \subseteq O(g(n)) \cap \Omega(g(n)) \tag{22}$$

Now you would need to do the same for the second (superset) part of the proof. You may find that this format is clearer, but it's a lot longer: remember that the above proof is just lines 1–5 of the recommended format. More experienced students will find that the additional text obscures, rather than clarifies, what's going on.