CS 350 Algorithms and Complexity

Winter 2019

Lecture 9: Divide & Conquer
The Master Theorem, Mergesort & Quicksort

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MergeSort

ALGORITHM *Mergesort*$(A[0..n-1])$

//Sorts array $A[0..n-1]$ by recursive mergesort
//Input: An array $A[0..n-1]$ of orderable elements
//Output: Array $A[0..n-1]$ sorted in nondecreasing order
if $n > 1$
    copy $A[0..\lfloor n/2 \rfloor - 1]$ to $B[0..\lfloor n/2 \rfloor - 1]$
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    Merge(B, C, A)

Merge

ALGORITHM *Merge*$(B[0..p-1], C[0..q-1], A[0..p+q-1])$

//Merges two sorted arrays into one sorted array
//Input: Arrays $B[0..p-1]$ and $C[0..q-1]$ both sorted
//Output: Sorted array $A[0..p+q-1]$ of the elements of B and C
$i \leftarrow 0;\ j \leftarrow 0;\ k \leftarrow 0$
while $i < p$ **and** $j < q$ **do**
    if $B[i] \leq C[j]$
        $A[k] \leftarrow B[i];\ i \leftarrow i+1$
    else $A[k] \leftarrow C[j];\ j \leftarrow j+1$
    $k \leftarrow k+1$
if $i = p$
    copy $C[j..q-1]$ to $A[k..p+q-1]$
else copy $B[i..p-1]$ to $A[k..p+q-1]$

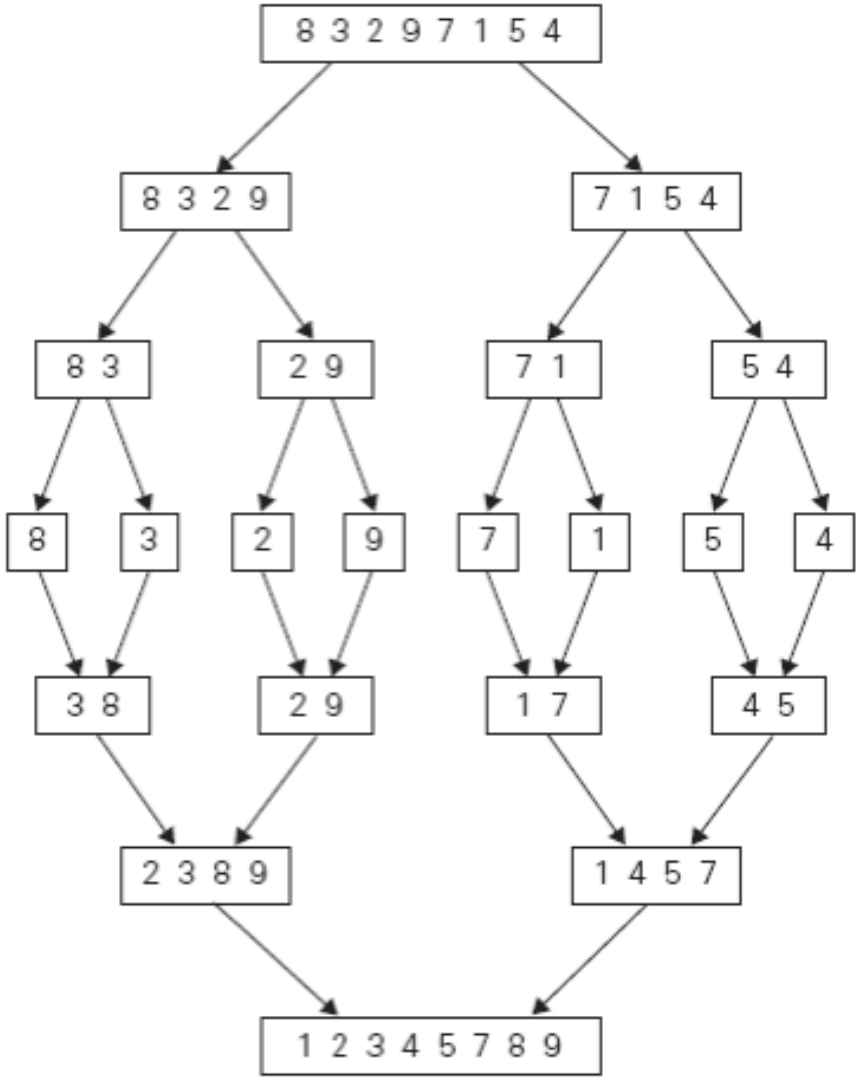

FIGURE 5.2 Example of mergesort operation.

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$$C(n)/n = C(1) \underbrace{+1+1+1+\ldots}_{\lg n \text{ terms}} \underbrace{- 1/n - 2/n - 4/n - \ldots}_{\lg n \text{ terms}}$$

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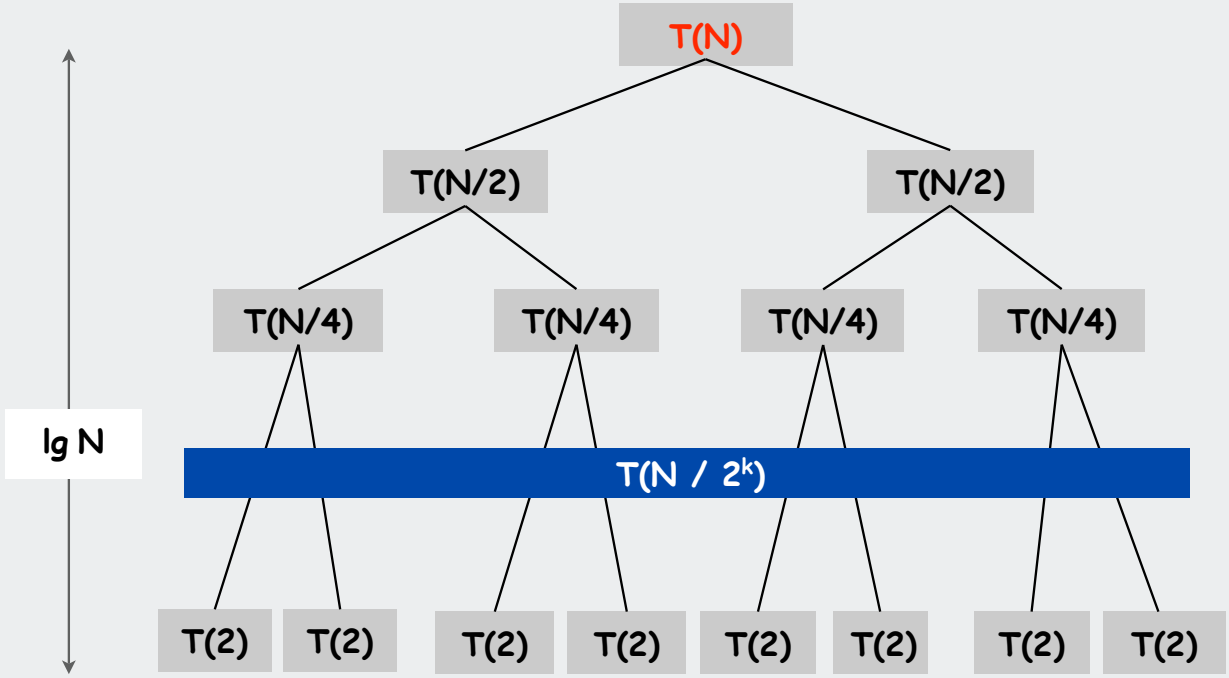
$$C(n) = n \lg n$$

Look at the recursion tree:

$T(N) = 2\,T(N/2) + N$

for $N > 1$, with $T(1) = 0$

(assume that N is a power of 2)



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10

from Sedgewick & Wayne http://www.cs.princeton.edu/courses/archive/spr07/cos226/lectures/04MergeQuick.pdf

Divide & Conquer

- When designing a divide and conquer algorithm, the time for the "split" and "combine" phases contributes to the cost:

  A. not at all
  B. in a small way
  C. in a major way
  D. in a way that dominates the total cost
  E. in a way that depends on the algorithm

Divide & Conquer

- In mergesort, the total cost is dominated by the cost of:

  A. the split phase (sub-array copy)
  B. the combine phase (merge)
  C. the recursion
  D. something else

Divide & Conquer

- In Quicksort, the total cost is dominated by the cost of:

  A. the split phase (partition)
  B. the combine phase (sub-array composition)
  C. the recursion
  D. something else

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- Suppose that $T_{combine}(n) \in \Theta(n^d)$ Then:

$$T(n) \in \begin{cases} \Theta(n^d) & \text{when } a < b^d \\ \Theta(n^d \log n) & \text{when } a = b^d \\ \Theta(n^{\log_b a}) & \text{when } a > b^d \end{cases}$$

Application to Mergesort:

- We divide the problem into 2 (roughly equal) parts, so $b = 2$
- We have to solve *both* of them, so $a = 2$
- Combining the parts is linear, i.e., in $\Theta(n^1)$, so $d = 1$
- Hence, in the Master Theorem, $a = 2 = 2^1 = b^d$

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So $T(n) = \Theta(n \log n)$

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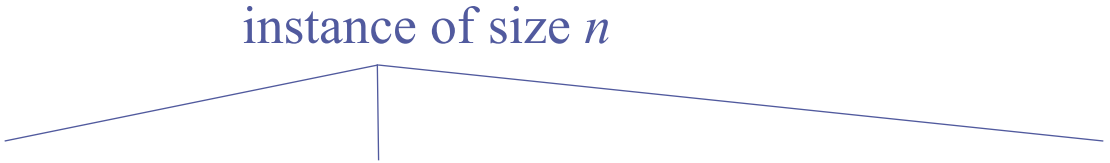

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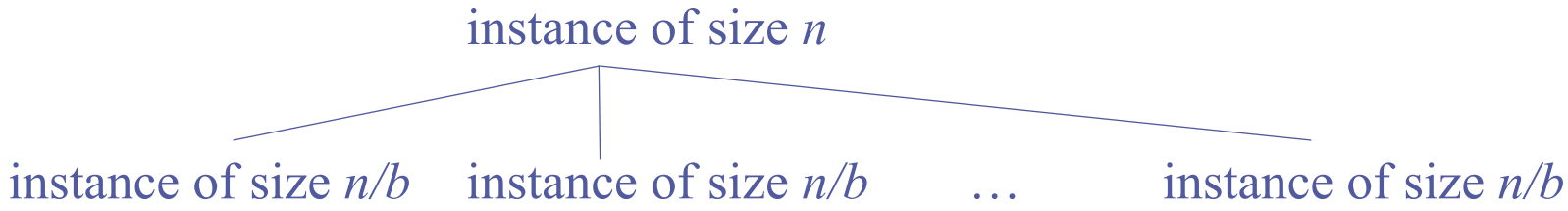

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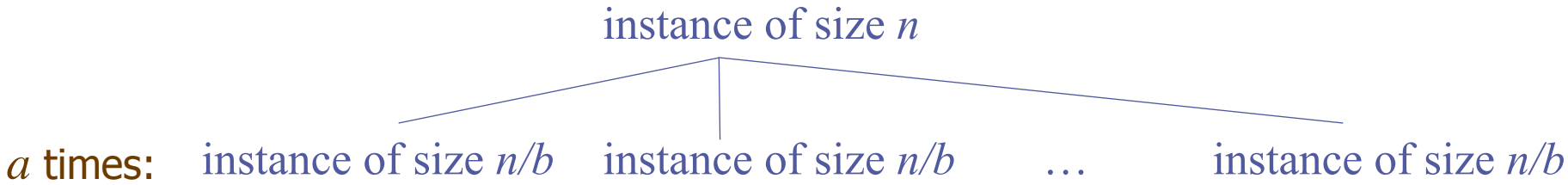

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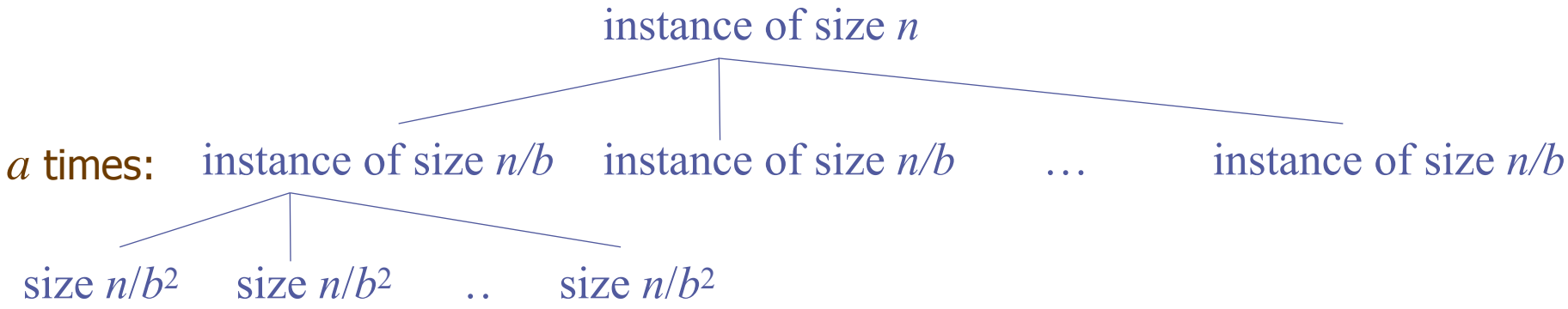

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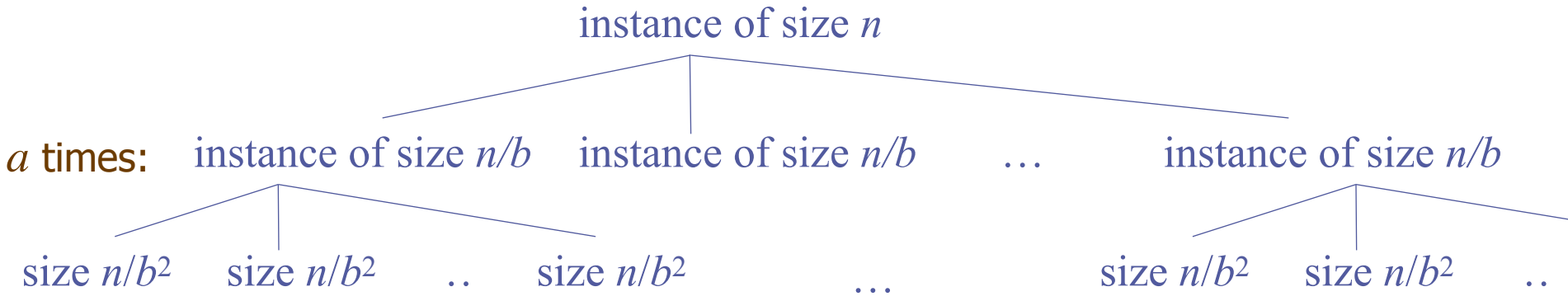

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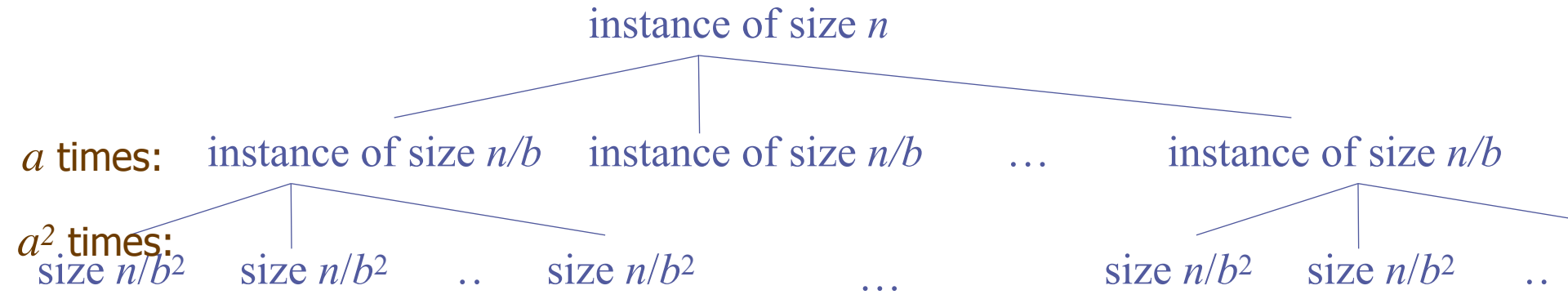

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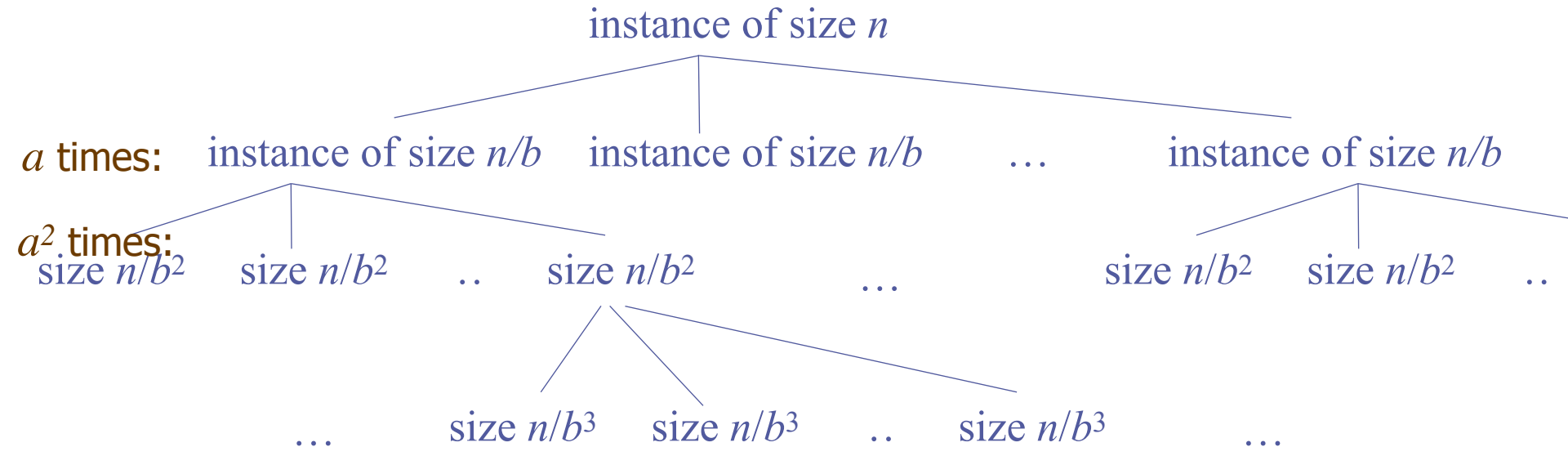

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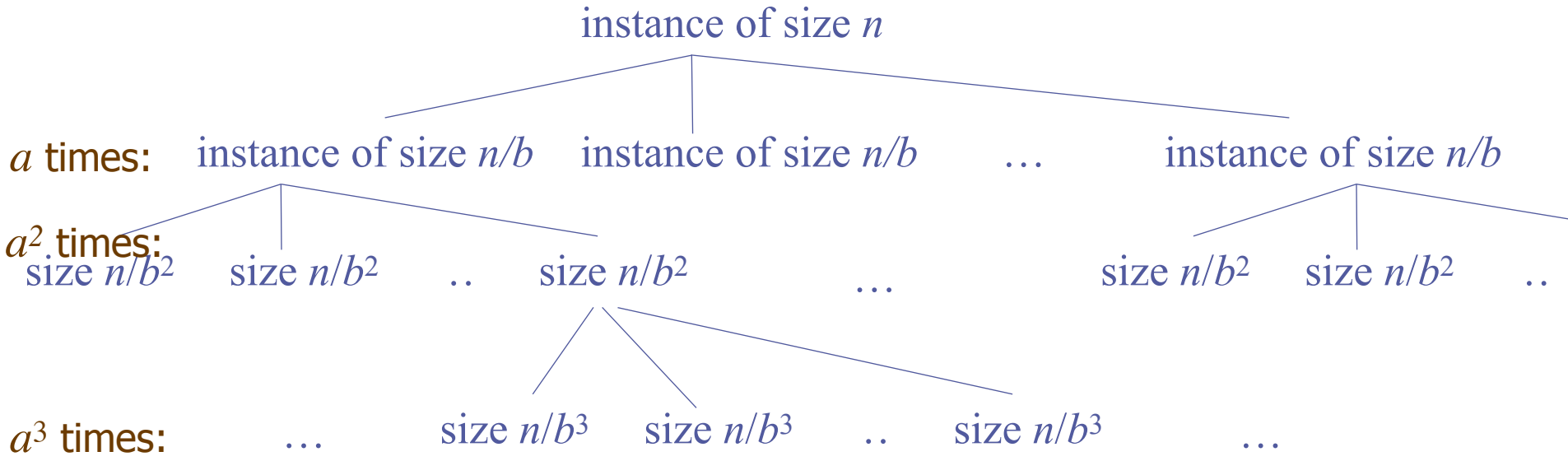

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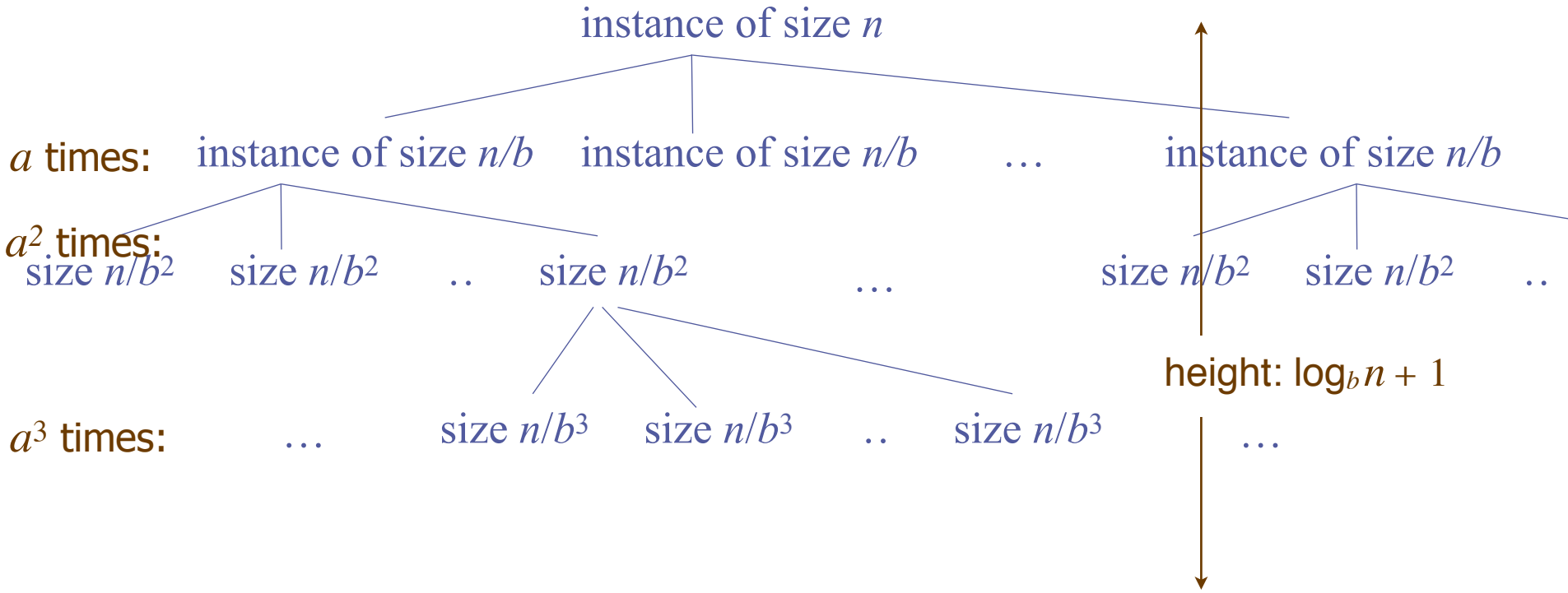

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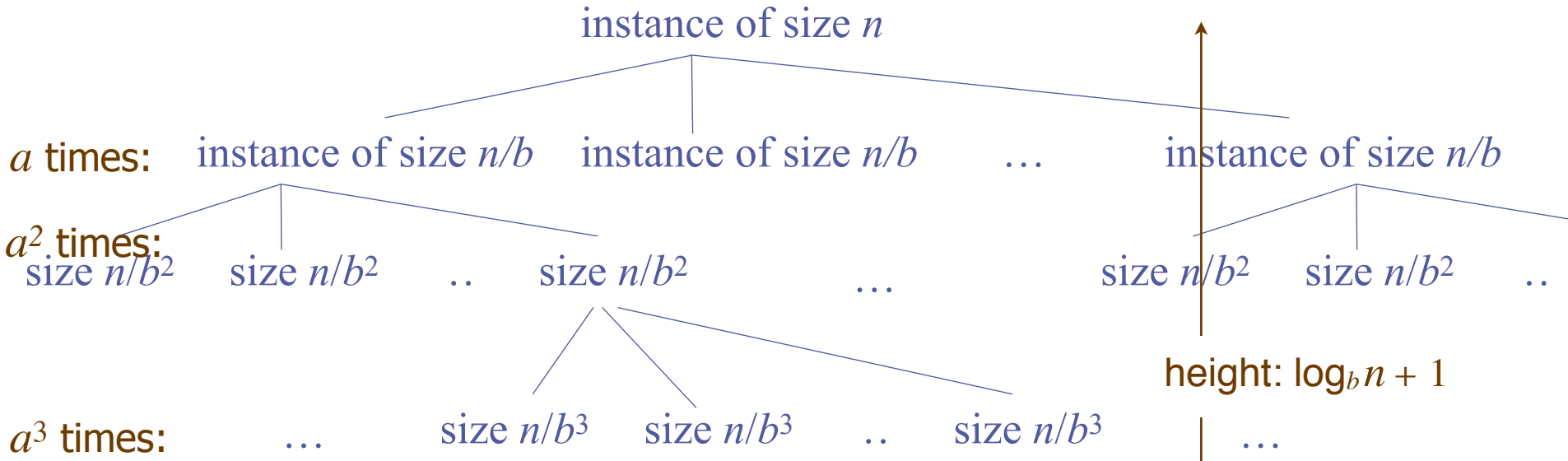

At level i: a^i problems of size $\frac{n}{b^i}$, *each* requiring combining work in $\Theta\left((\frac{n}{b^i})^d\right)$

So total work is at level i is: $\Theta\left(a^i\left(\frac{n}{b^i}\right)^d\right) = \Theta\left(n^d\left(\frac{a}{b^d}\right)^i\right)$

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$$\sum_{i=0}^{n} a^i = \frac{a^{n+1} - 1}{a - 1} = \frac{1 - a^{n+1}}{1 - a} \quad (a \neq 1)$$

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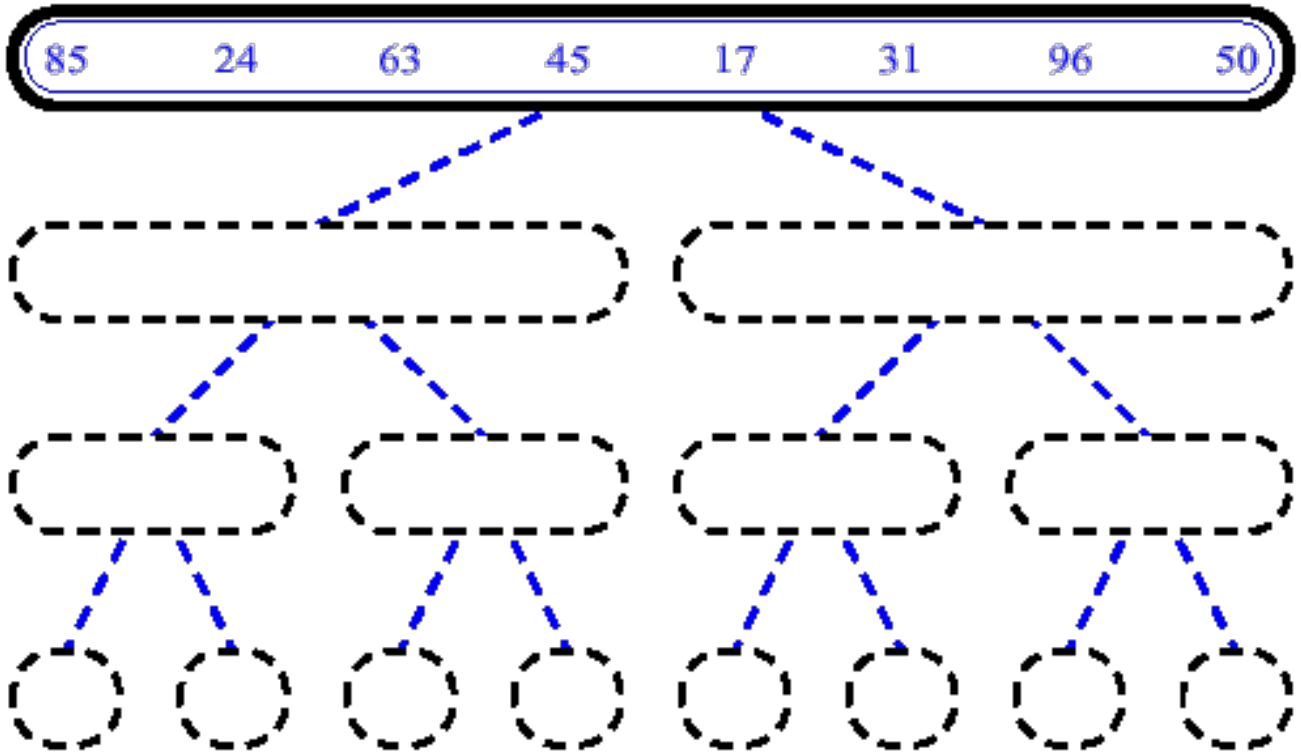
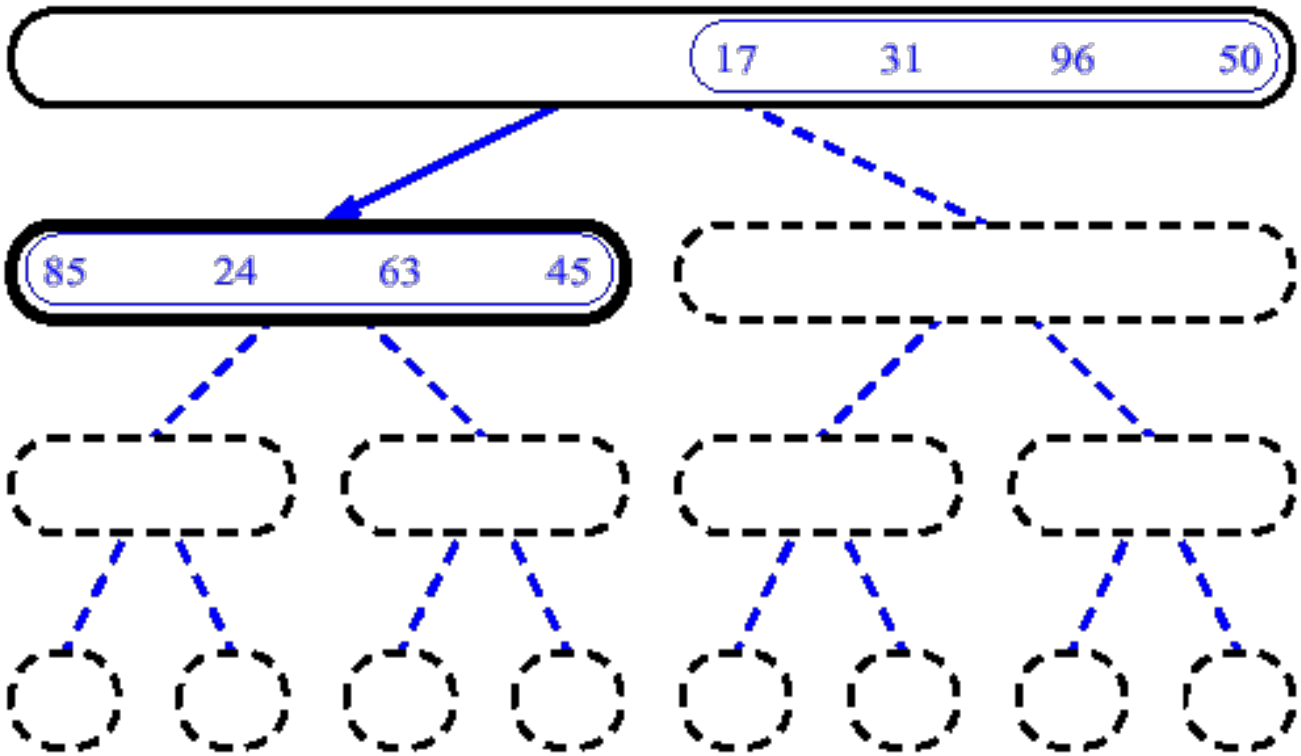
MergeSort (Example) — 1

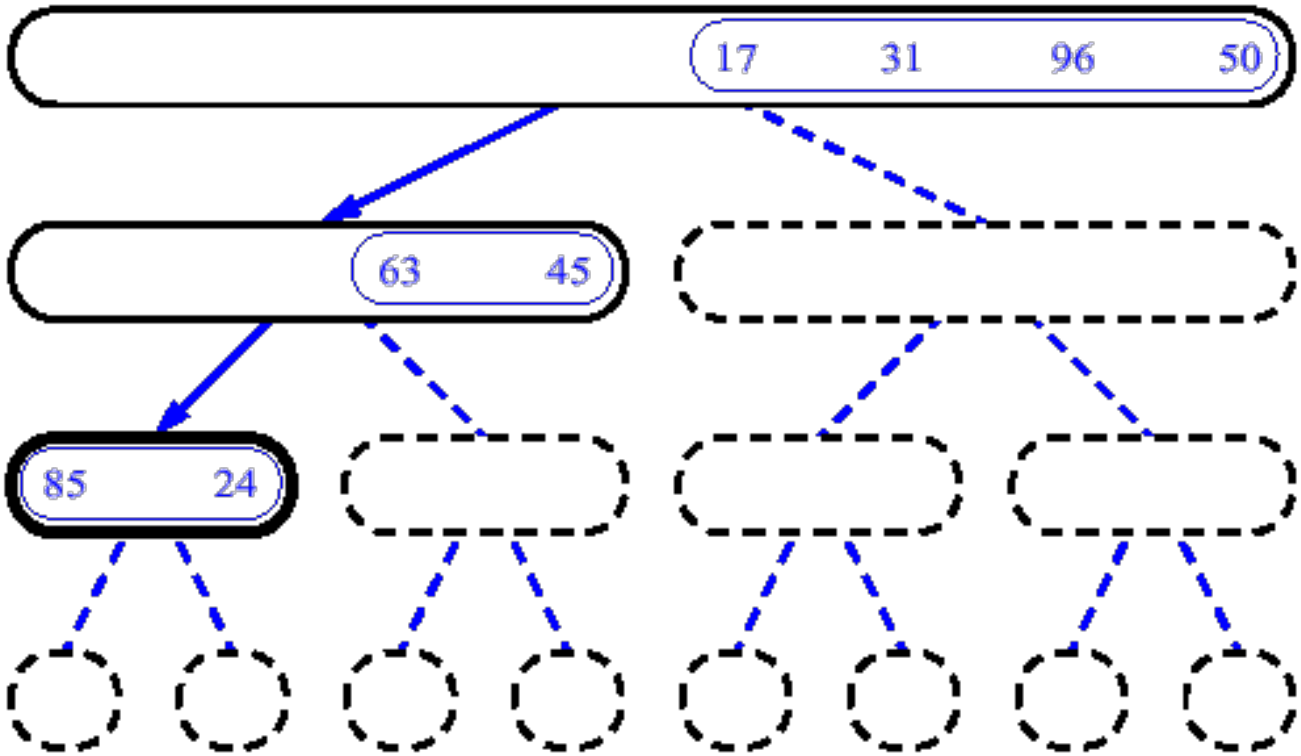


Example from Ján Maňuch, Simon Fraser University

MergeSort (Example) — 2

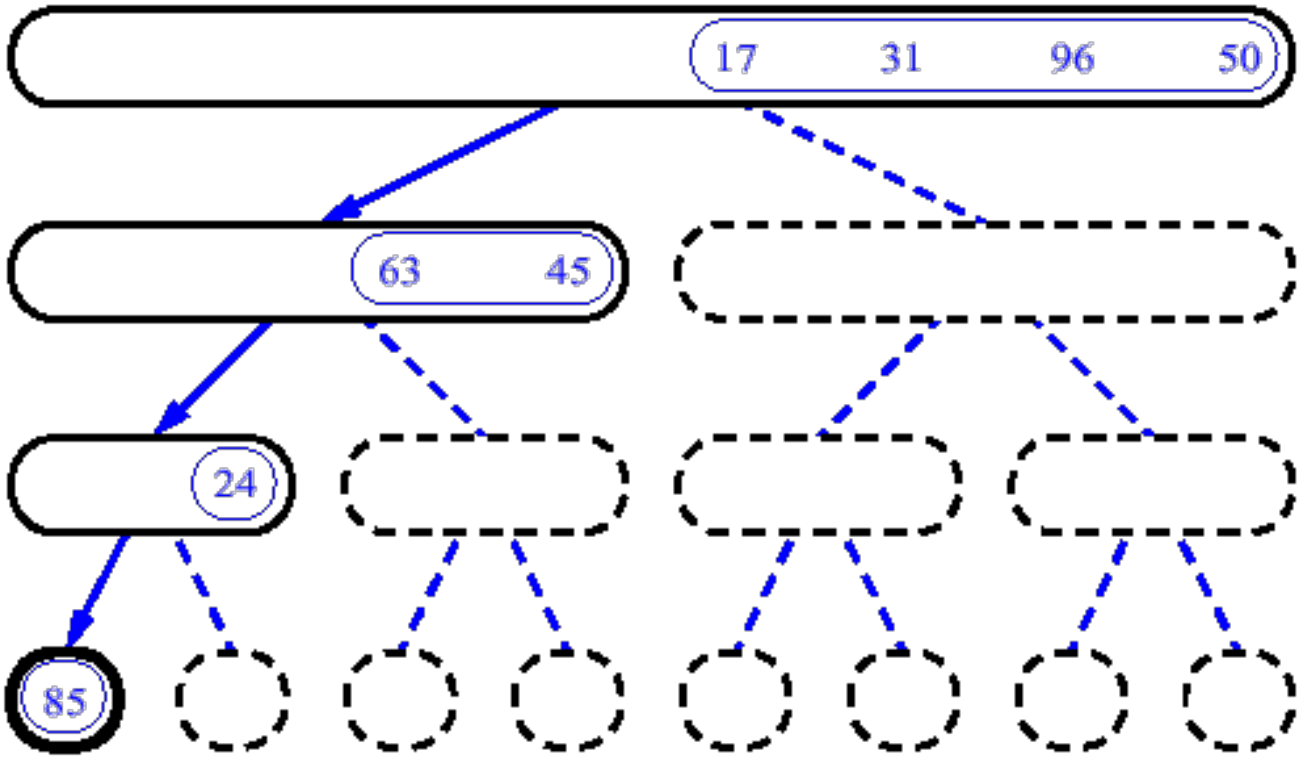


MergeSort (Example) — 3



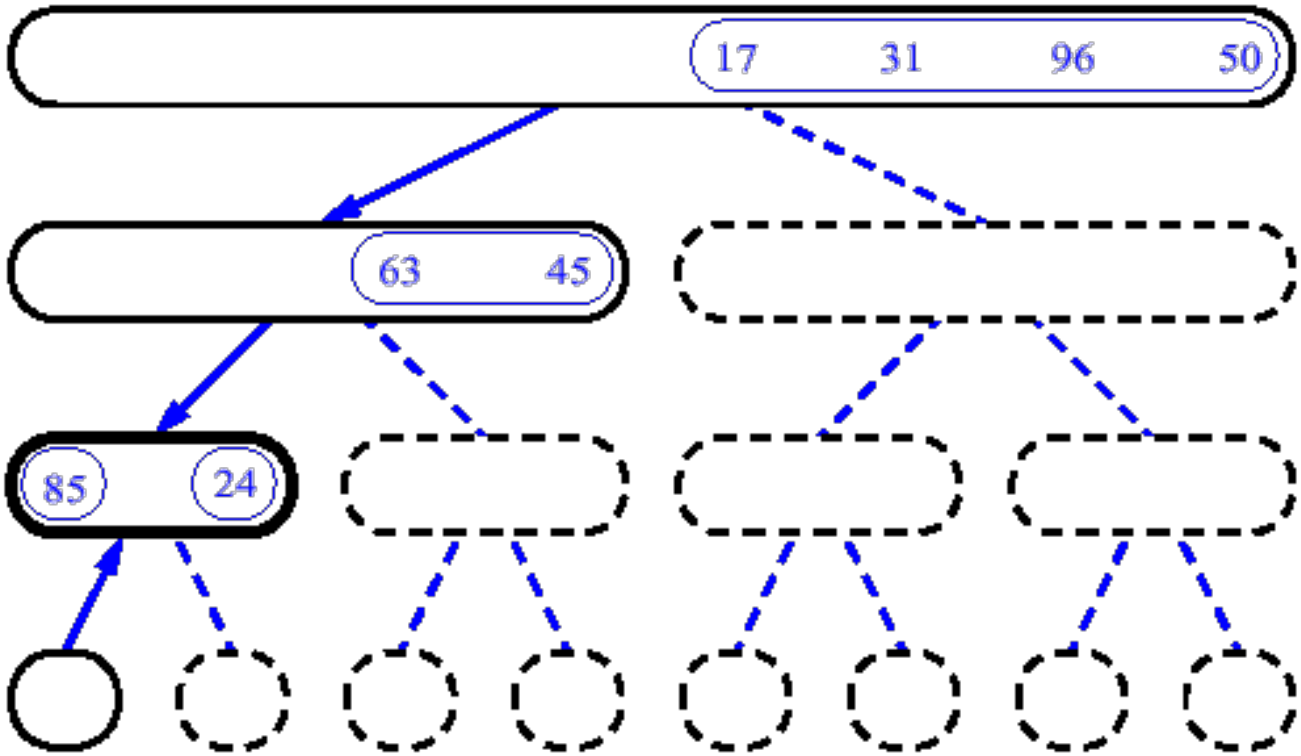
Example from Ján Maňuch, Simon Fraser University

MergeSort (Example) — 4



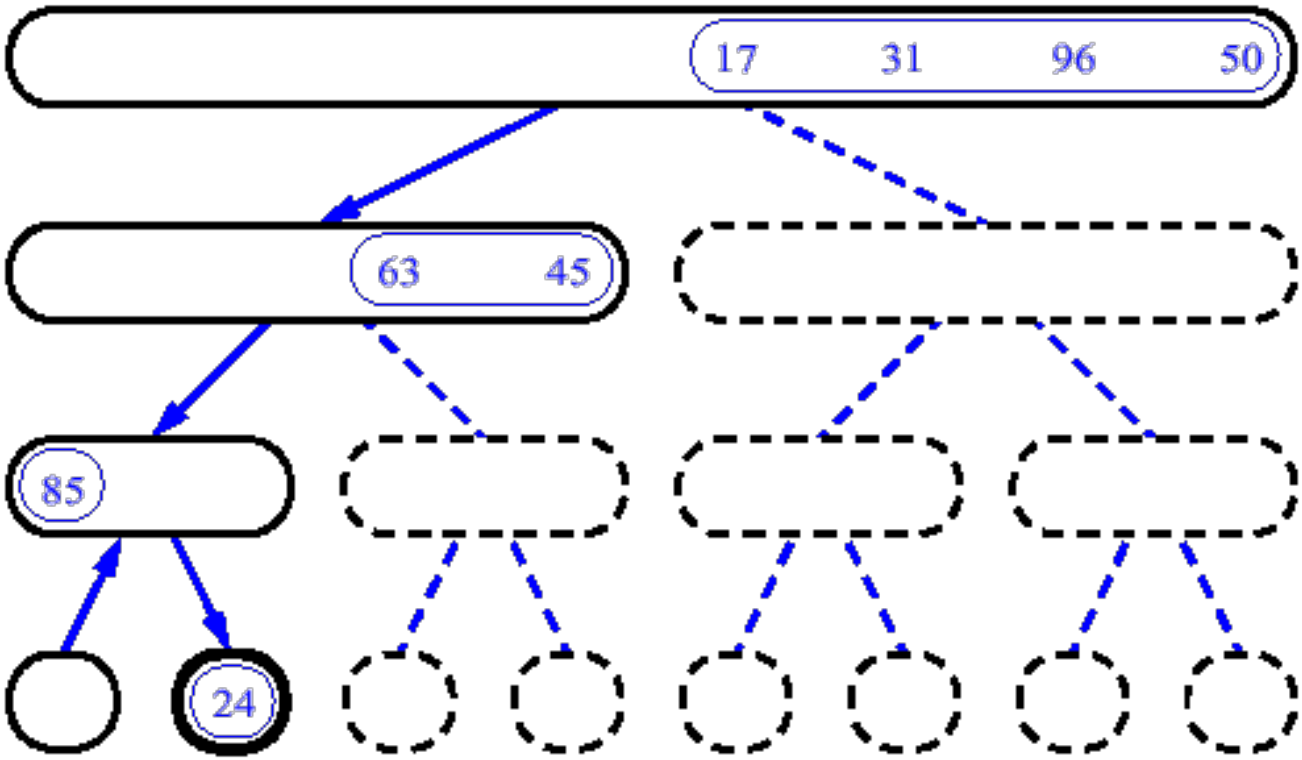
Example from Ján Maňuch, Simon Fraser University

MergeSort (Example) — 5



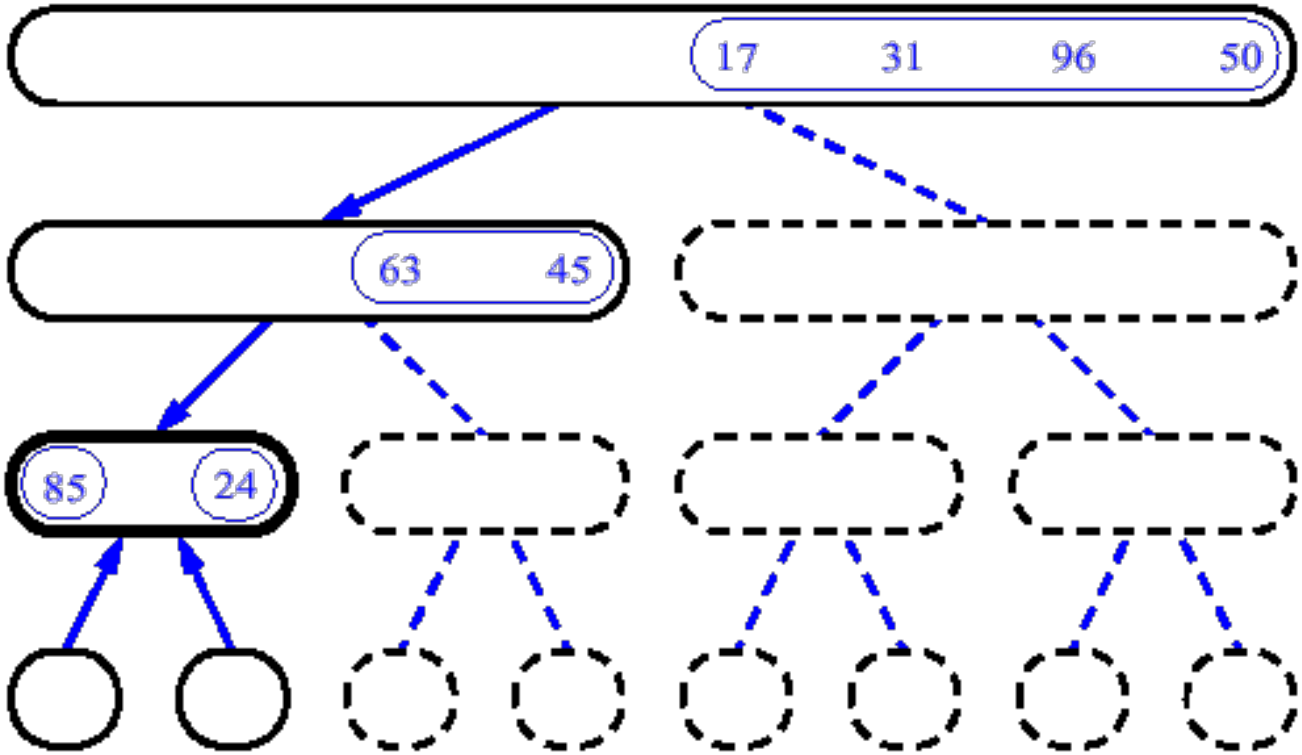
Example from Ján Maňuch, Simon Fraser University

MergeSort (Example) — 6



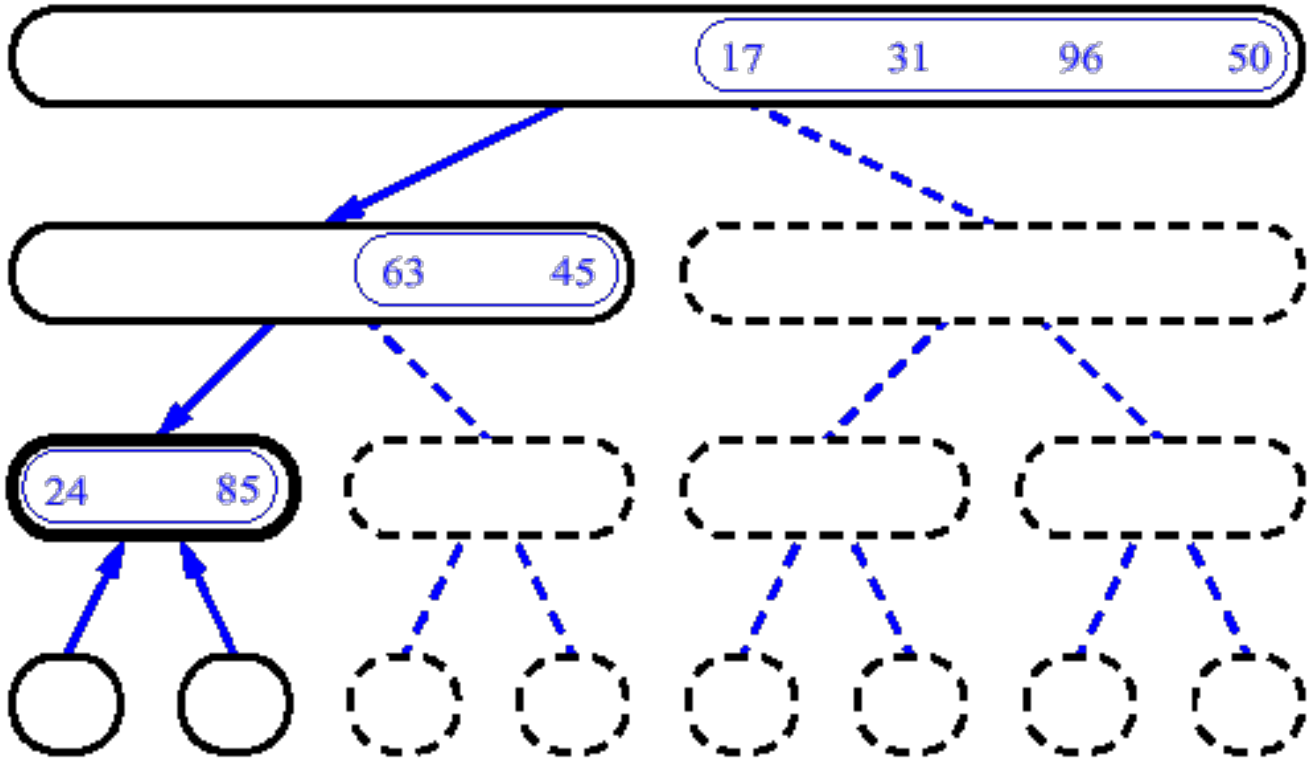
Example from Ján Maňuch, Simon Fraser University

MergeSort (Example) — 7



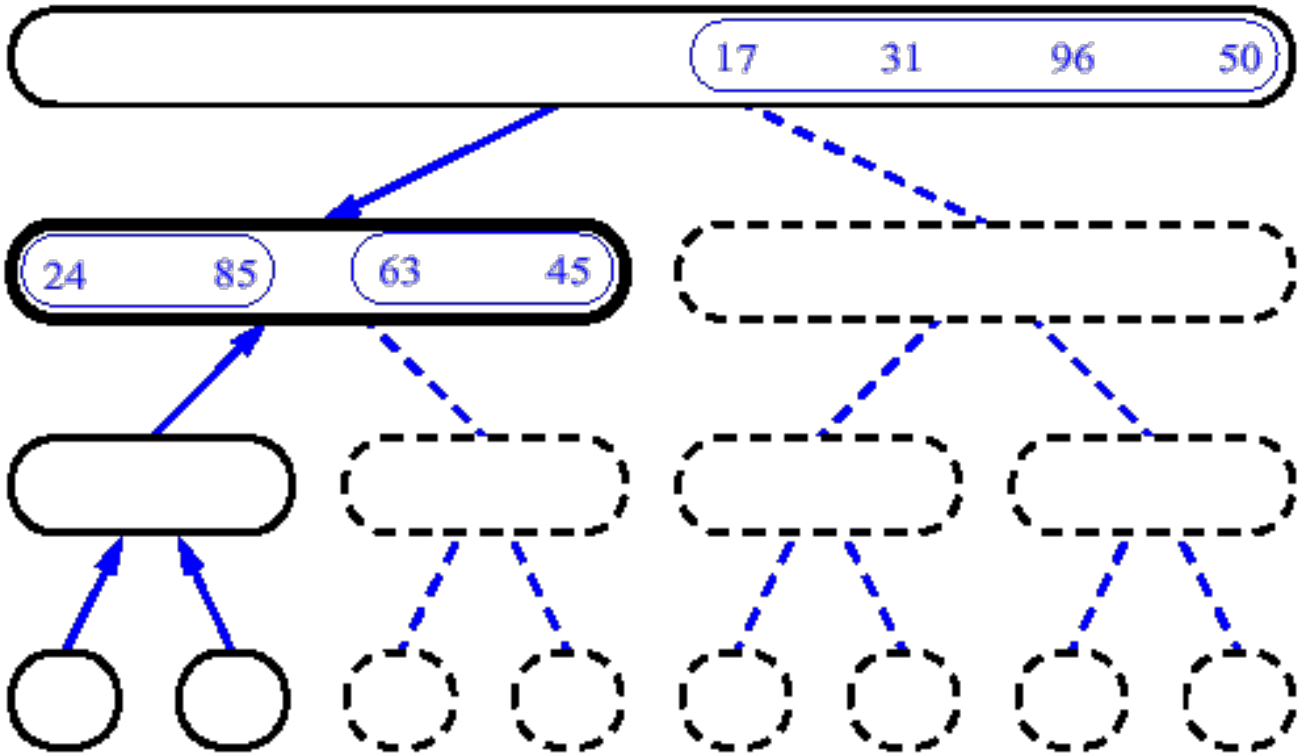
Example from Ján Maňuch, Simon Fraser University

MergeSort (Example) — 8



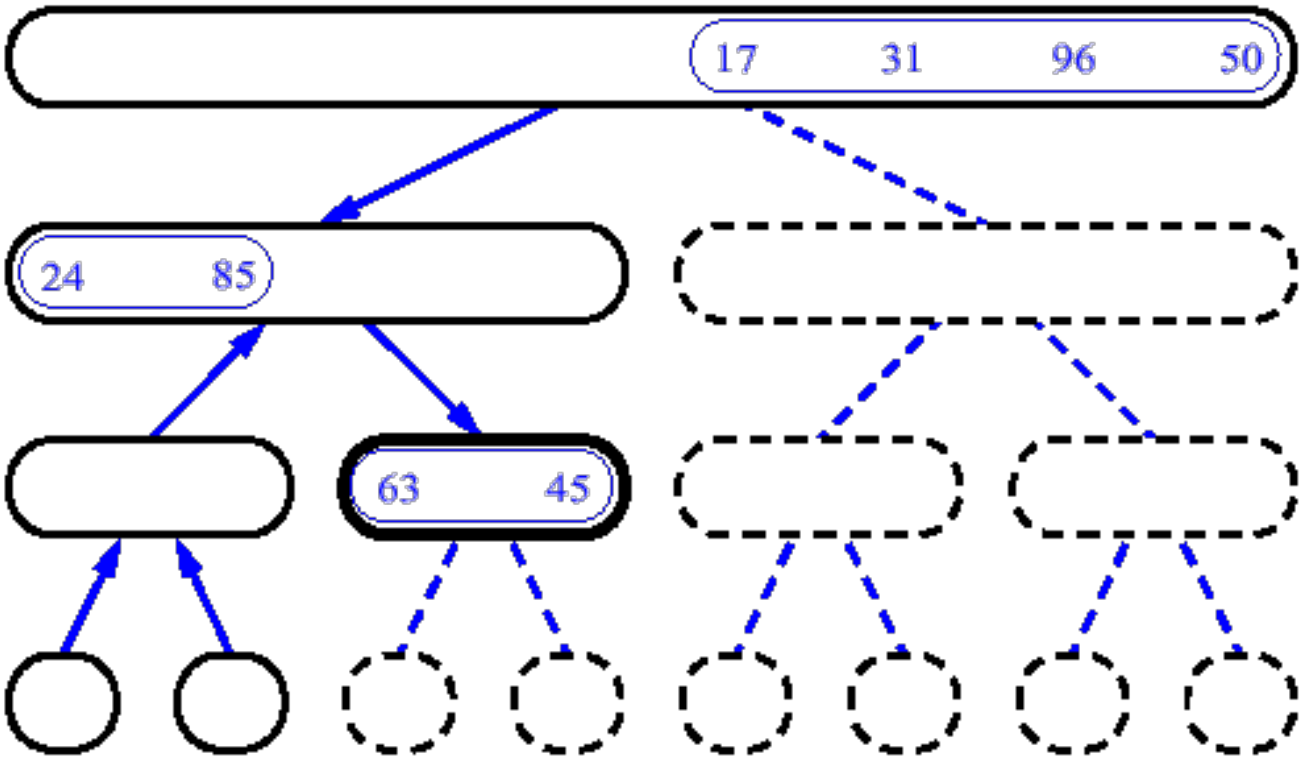
Example from Ján Maňuch, Simon Fraser University

MergeSort (Example) — 9



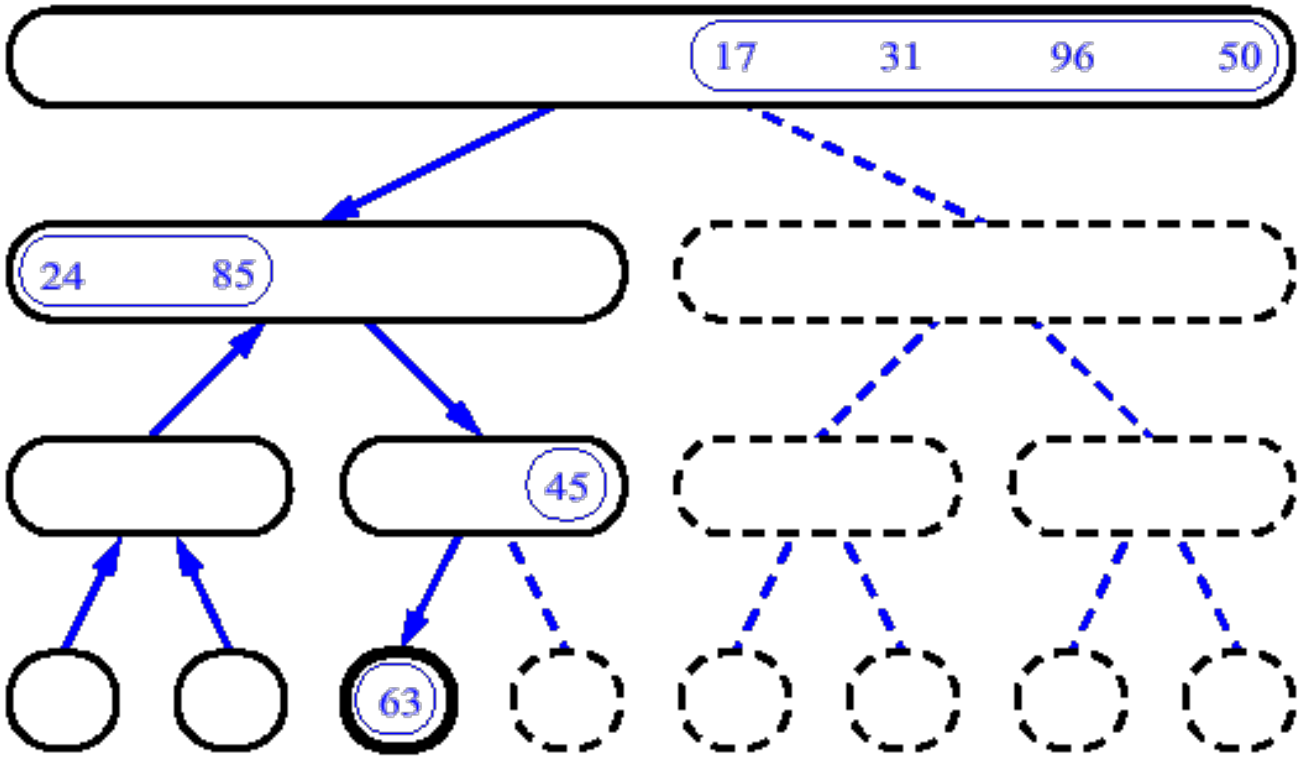
Example from Ján Maňuch, Simon Fraser University

MergeSort (Example) — 10



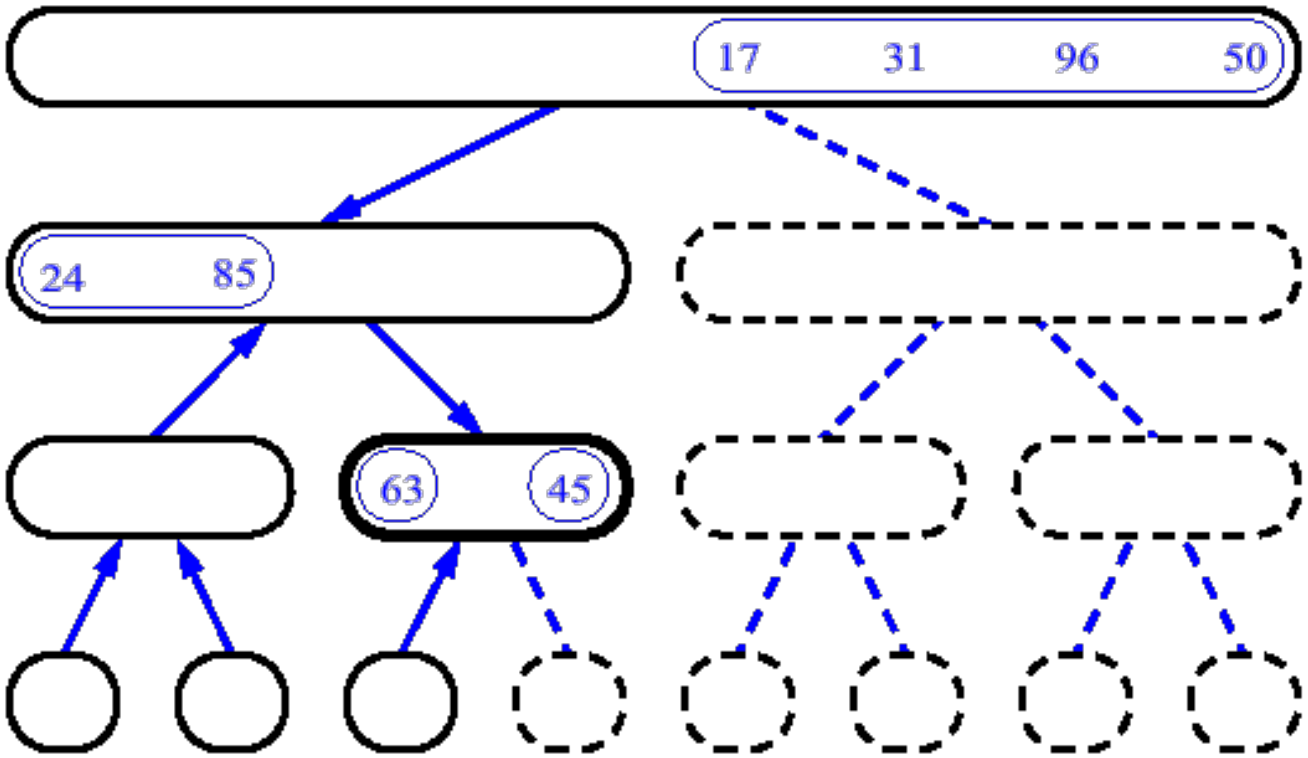
Example from Ján Maňuch, Simon Fraser University

MergeSort (Example) — 11



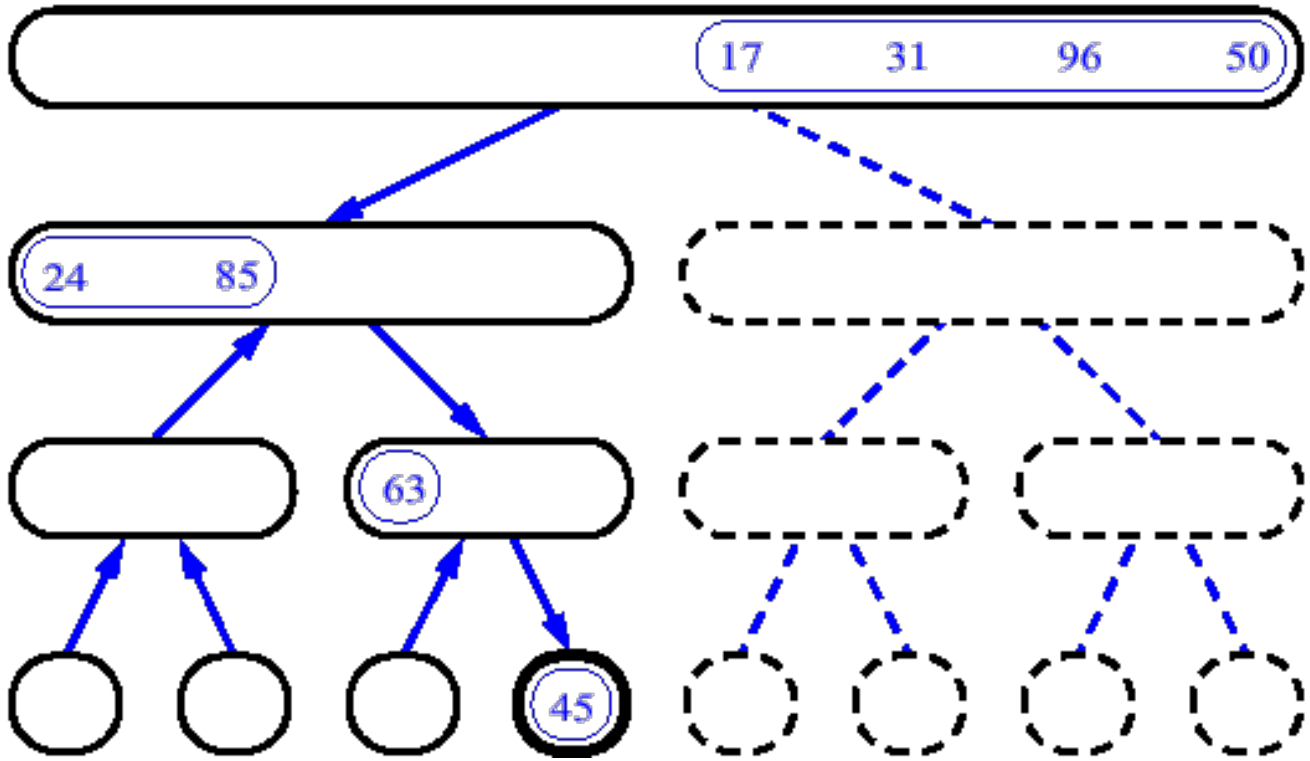
Example from Ján Maňuch, Simon Fraser University

MergeSort (Example) — 12



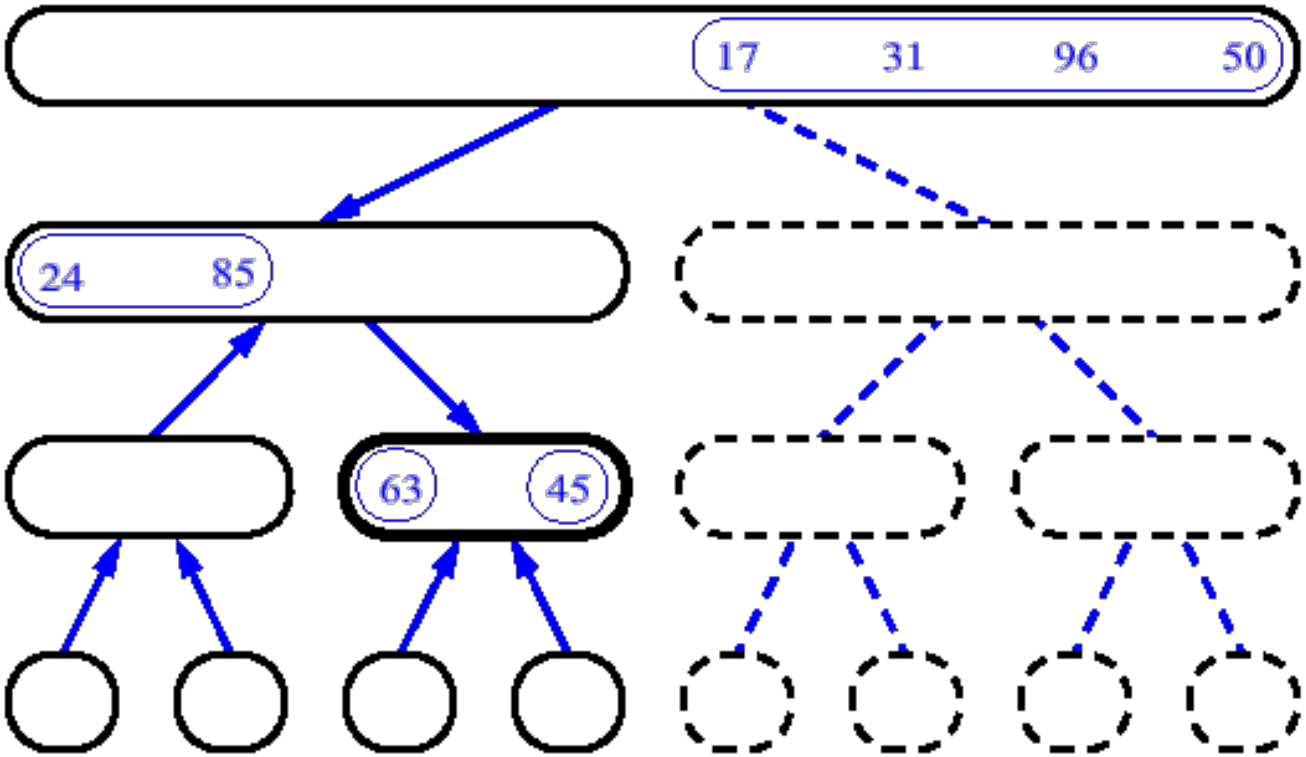
Example from Ján Maňuch, Simon Fraser University

MergeSort (Example) — 13



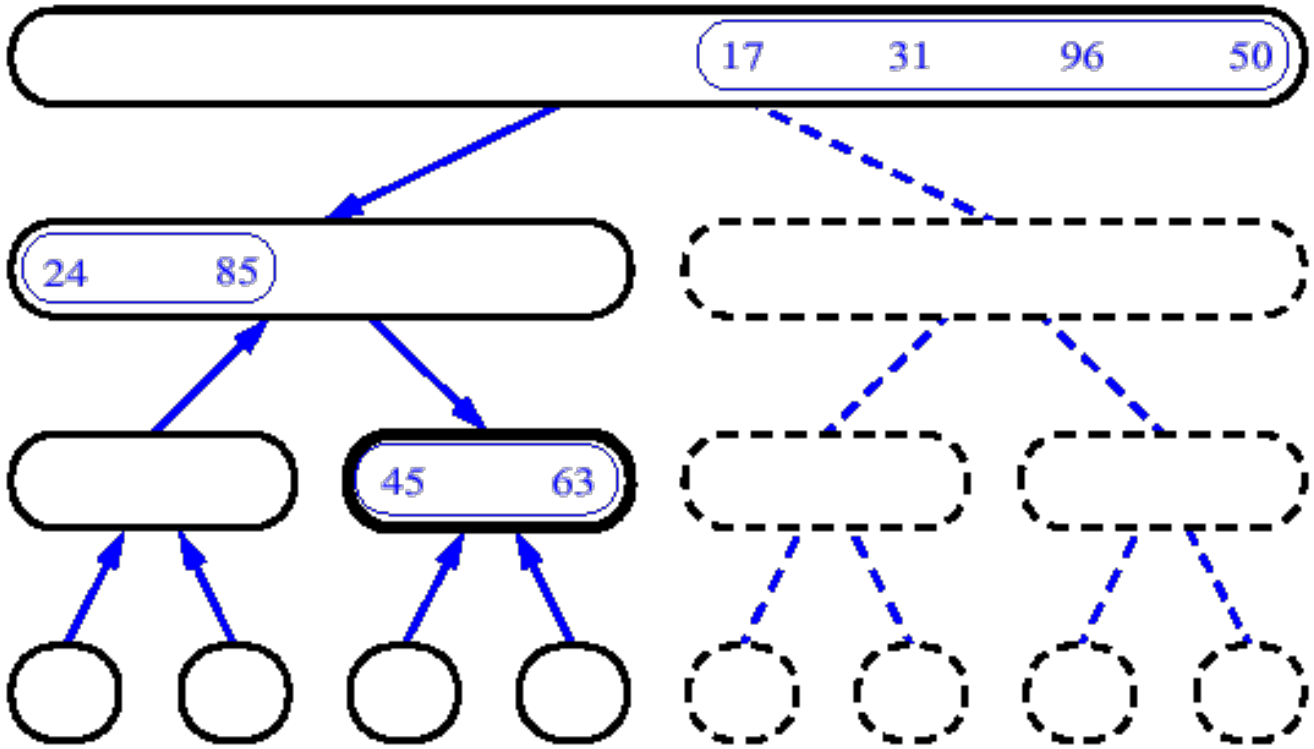
Example from Ján Maňuch, Simon Fraser University

MergeSort (Example) — 14



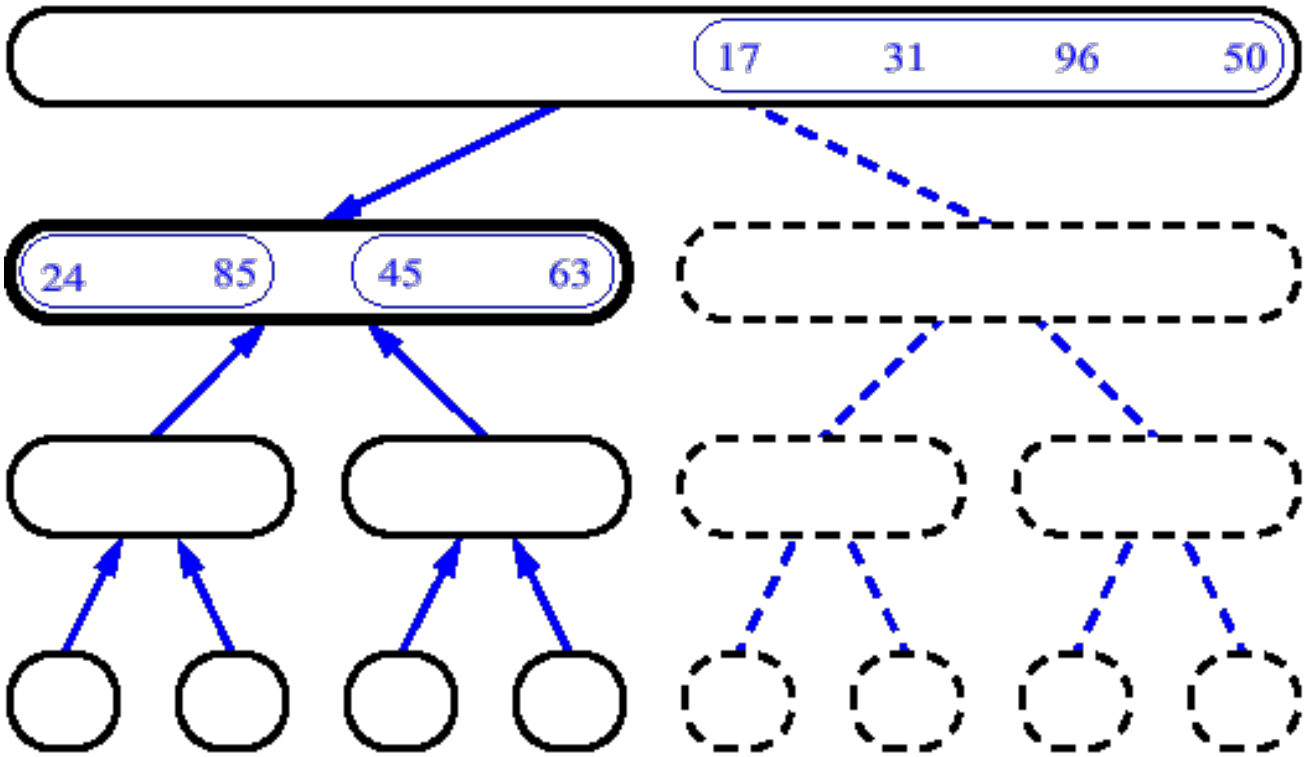
Example from Ján Maňuch, Simon Fraser University

MergeSort (Example) — 15



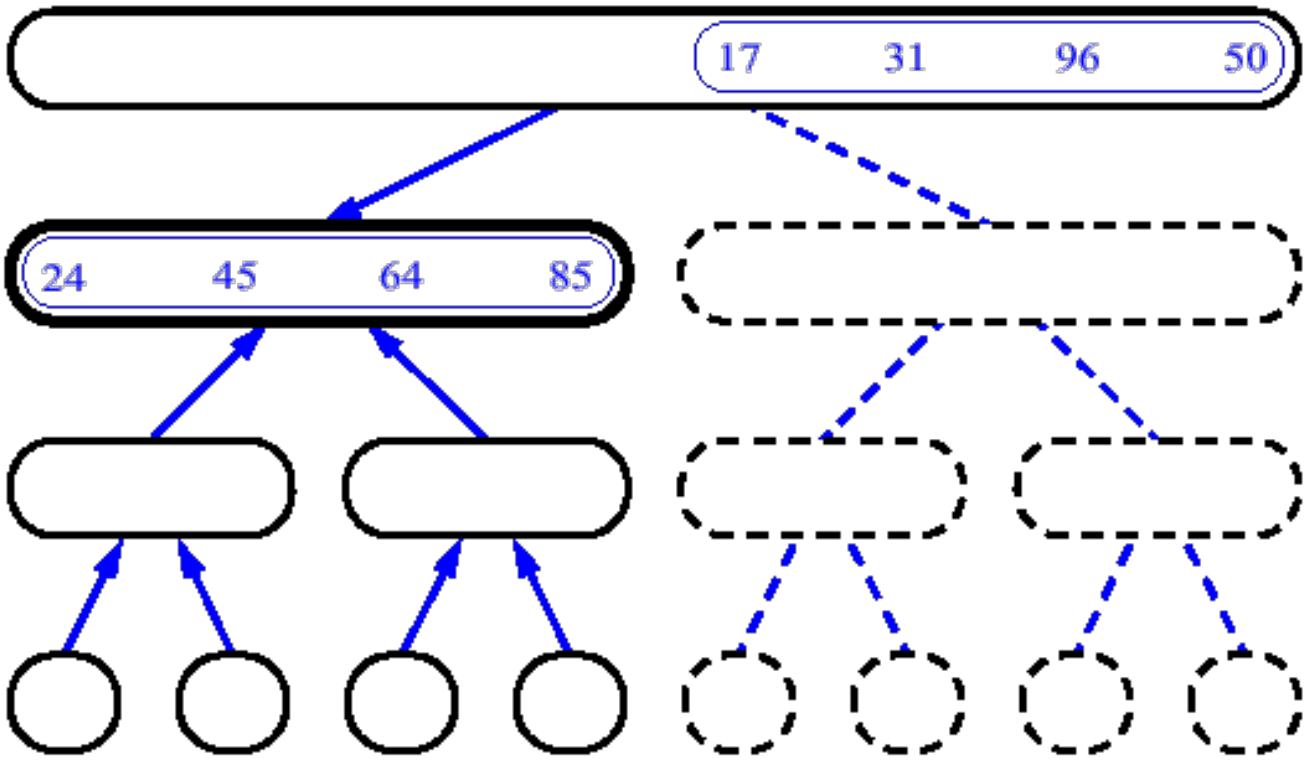
Example from Ján Maňuch, Simon Fraser University

MergeSort (Example) — 16



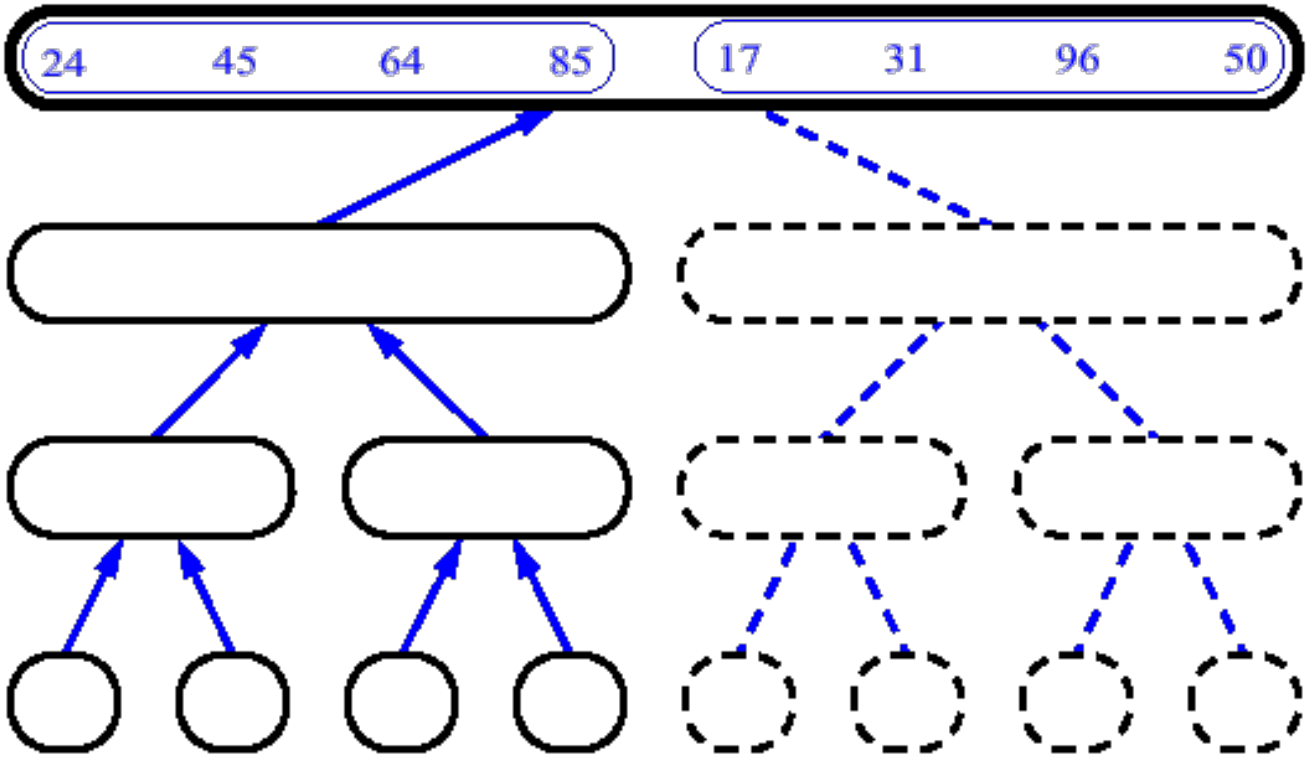
Example from Ján Maňuch, Simon Fraser University

MergeSort (Example) — 17



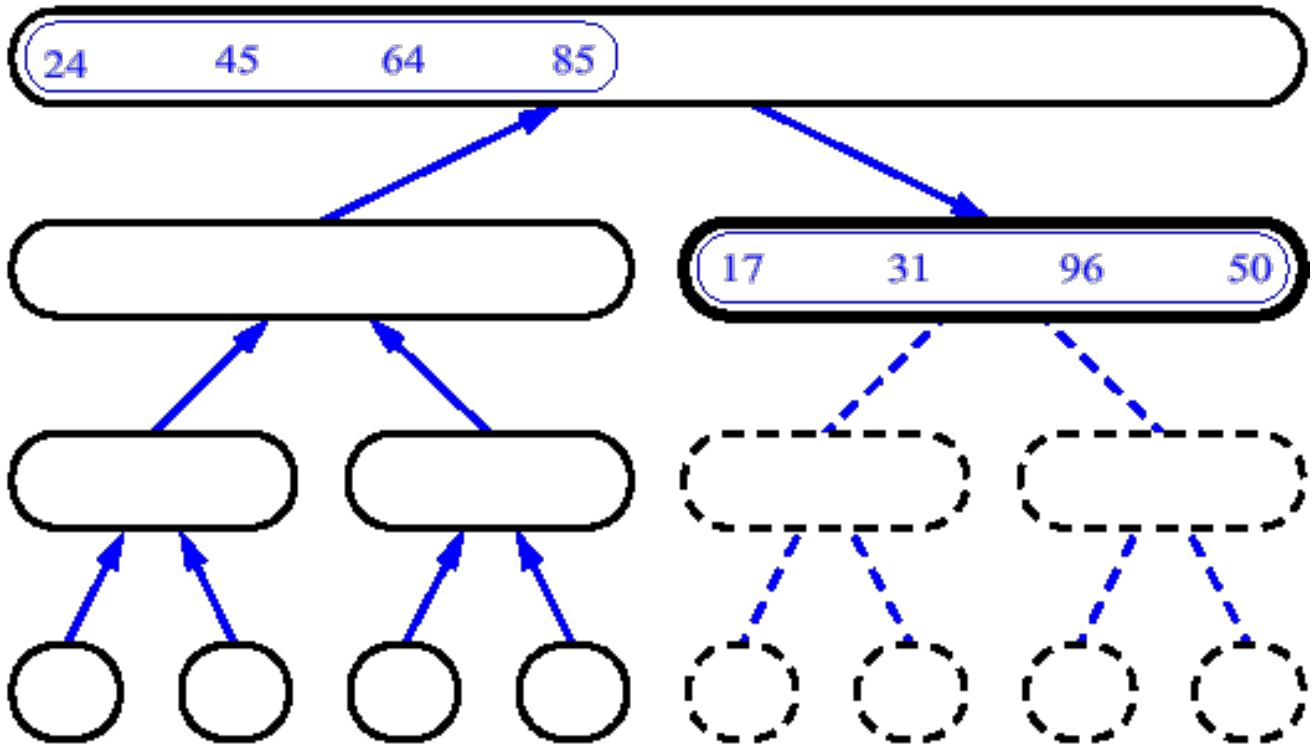
Example from Ján Maňuch, Simon Fraser University

MergeSort (Example) — 18



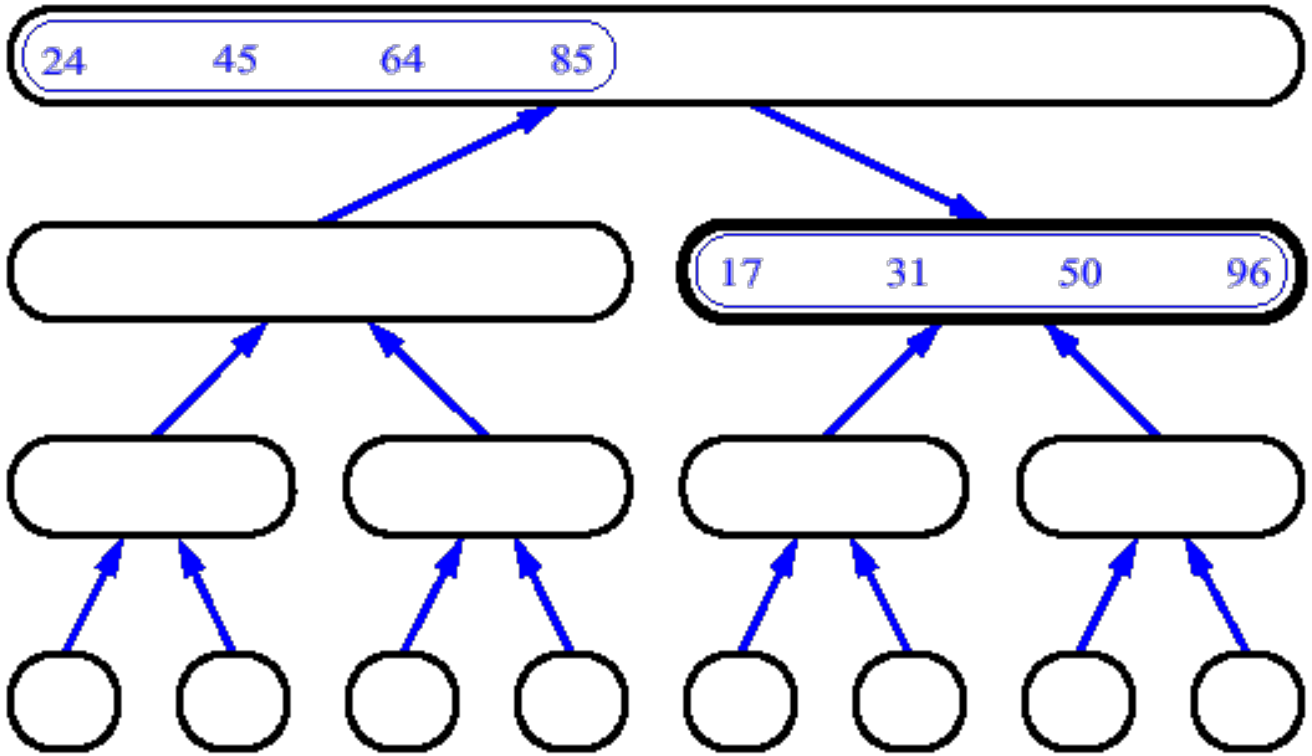
Example from Ján Maňuch, Simon Fraser University

MergeSort (Example) — 19



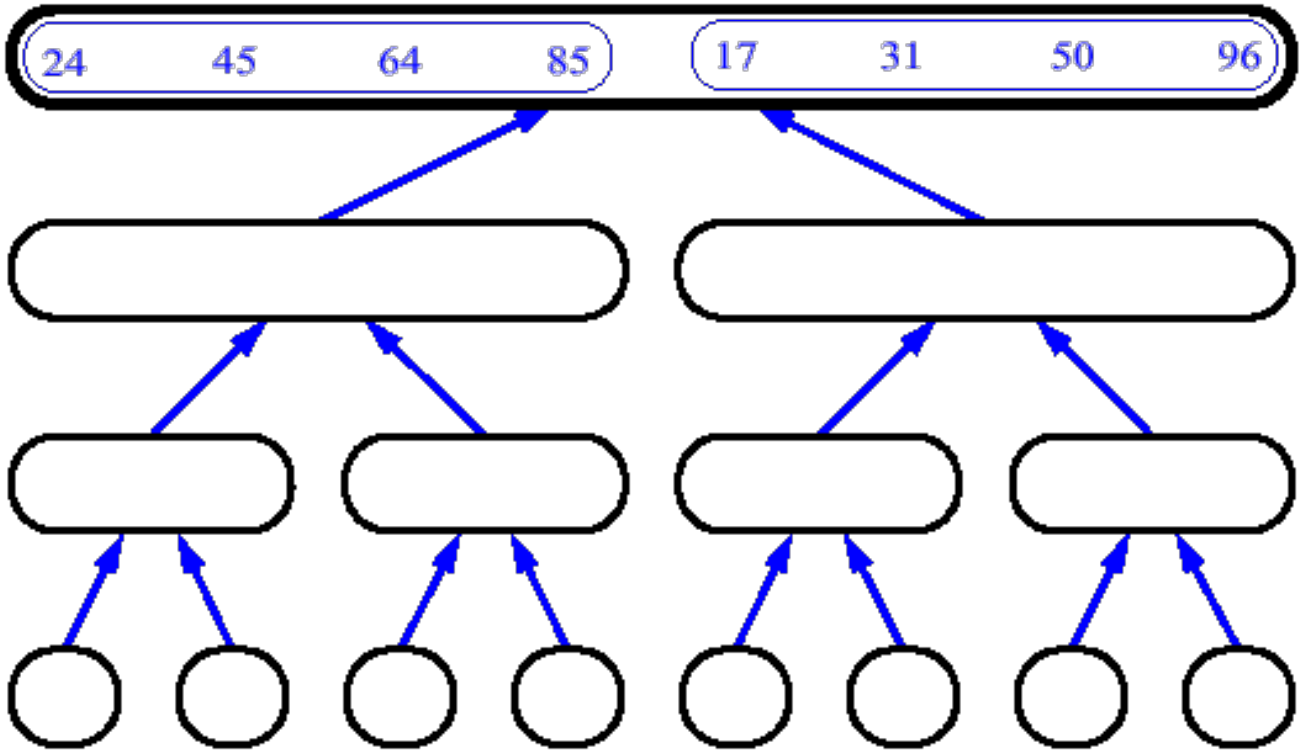
Example from Ján Maňuch, Simon Fraser University

MergeSort (Example) — 20



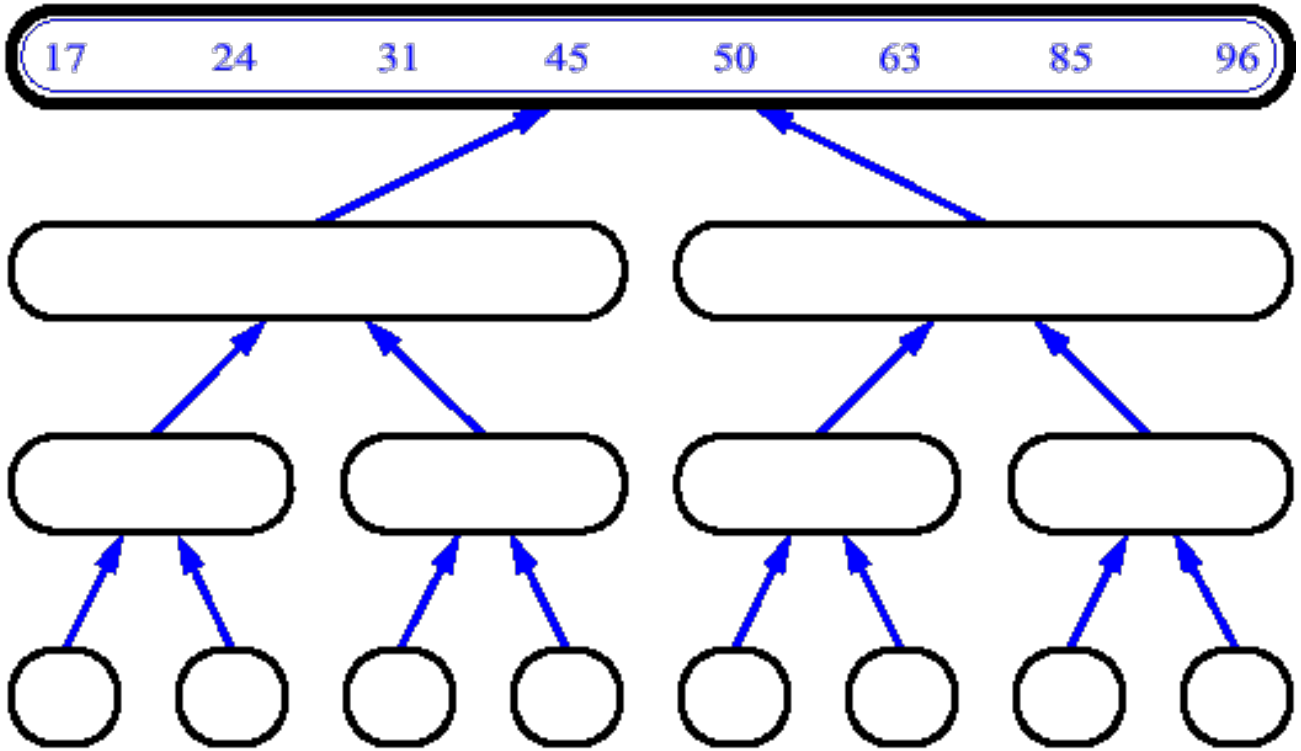
Example from Ján Maňuch, Simon Fraser University

MergeSort (Example) — 21



Example from Ján Maňuch, Simon Fraser University

MergeSort (Example) — 22



Example from Ján Maňuch, Simon Fraser University

Merge Phase (Example)

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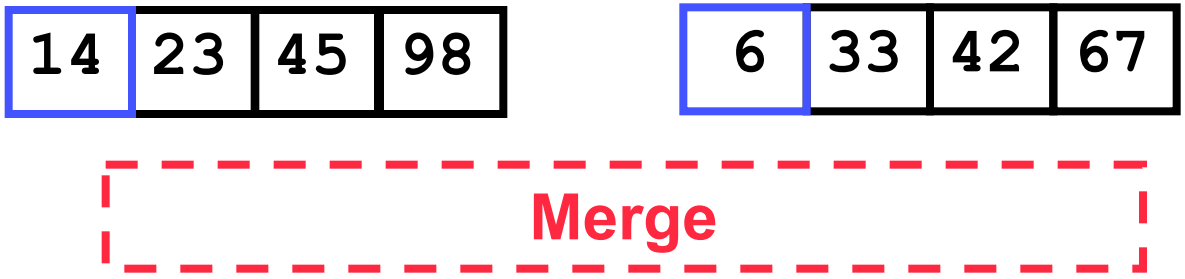

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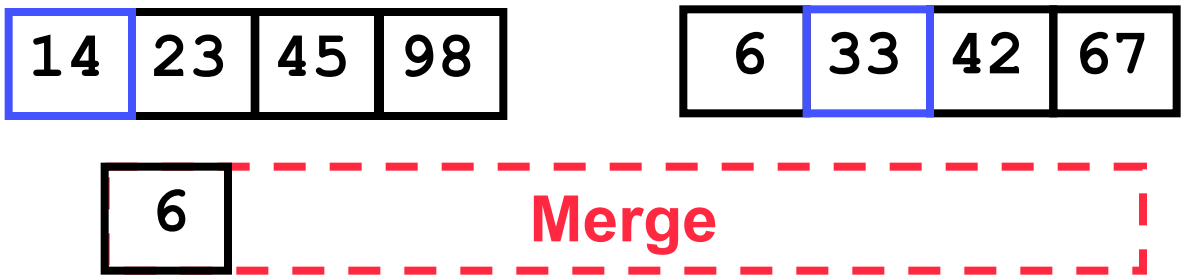

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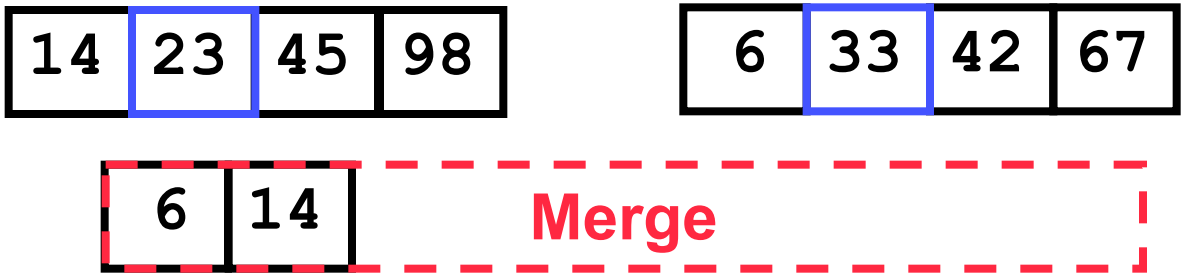

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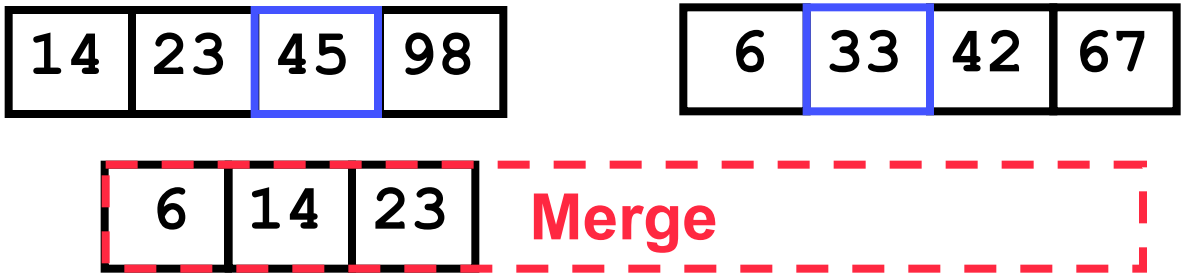

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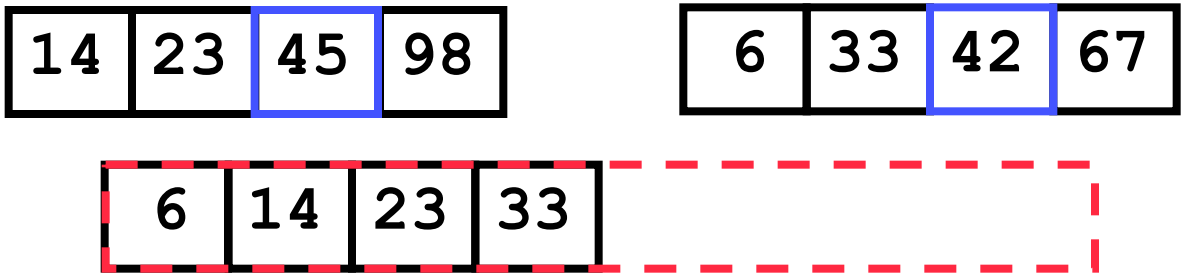

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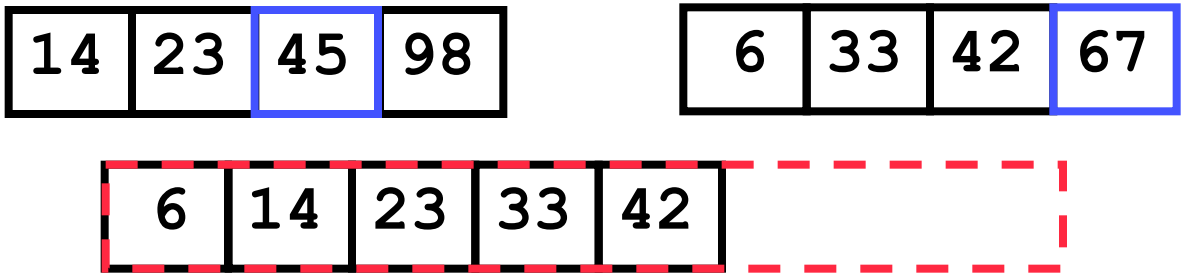

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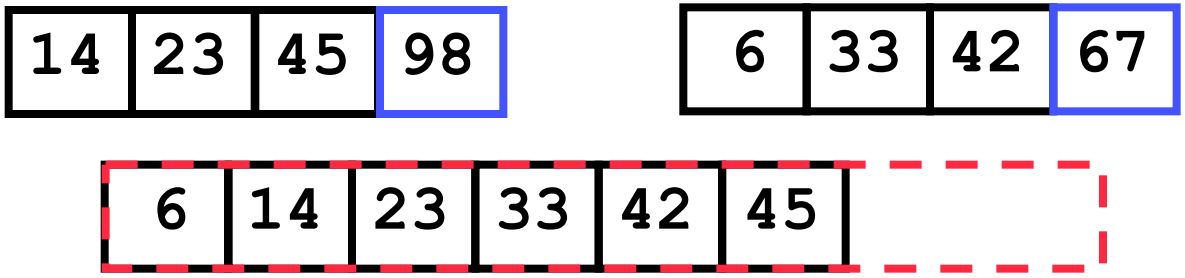

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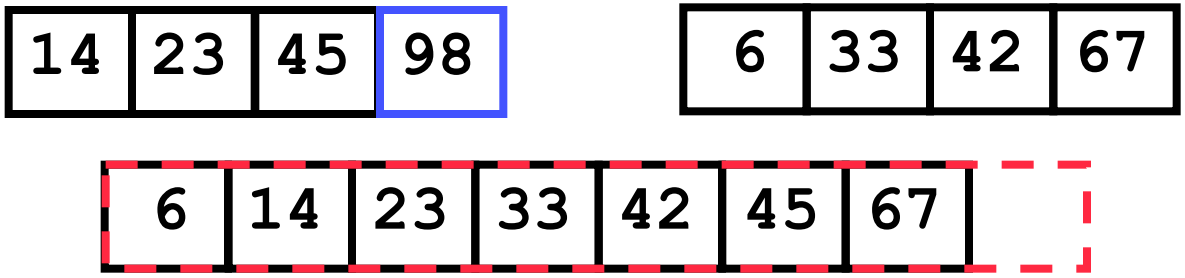

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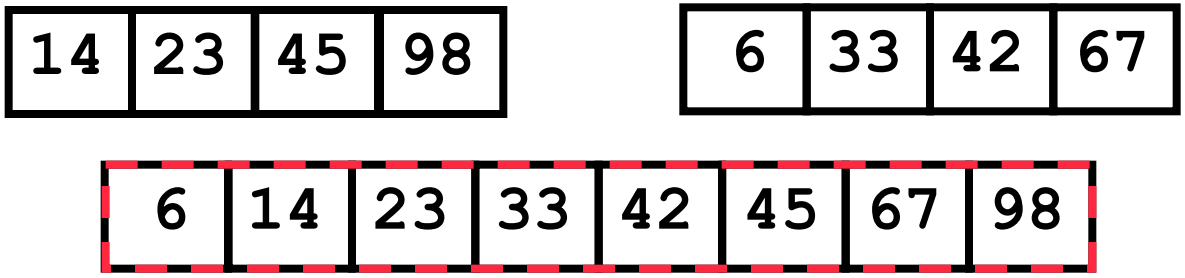


Example from Ján Maňuch, Simon Fraser University

What's wrong with this Pseudocode?

ALGORITHM *Mergesort*$(A[0..n-1])$

//Sorts array $A[0..n-1]$ by recursive mergesort
//Input: An array $A[0..n-1]$ of orderable elements
//Output: Array $A[0..n-1]$ sorted in nondecreasing order
if $n > 1$
    copy $A[0..\lfloor n/2 \rfloor - 1]$ to $B[0..\lfloor n/2 \rfloor - 1]$
    copy $A[\lfloor n/2 \rfloor..n-1]$ to $C[0..\lceil n/2 \rceil - 1]$
    Mergesort$(B[0..\lfloor n/2 \rfloor - 1])$
    Mergesort$(C[0..\lceil n/2 \rceil - 1])$
    Merge(B, C, A)

ALGORITHM *Merge*$(B[0..p-1], C[0..q-1], A[0..p+q-1])$

//Merges two sorted arrays into one sorted array
//Input: Arrays $B[0..p-1]$ and $C[0..q-1]$ both sorted
//Output: Sorted array $A[0..p+q-1]$ of the elements of B and C
$i \leftarrow 0;\ j \leftarrow 0;\ k \leftarrow 0$
while $i < p$ **and** $j < q$ **do**
    if $B[i] \le C[j]$
        $A[k] \leftarrow B[i];\ i \leftarrow i + 1$
    else $A[k] \leftarrow C[j];\ j \leftarrow j + 1$
    $k \leftarrow k + 1$
if $i = p$
    copy $C[j..q-1]$ to $A[k..p+q-1]$
else copy $B[i..p-1]$ to $A[k..p+q-1]$

```
1
2  /* Java program for Merge Sort from GeeksForGeeks.com */
3  class MergeSort
4  {
5      void merge(int arr[], int l, int m, int r) {
6          // Merges two subarrays of arr[].
7          // First subarray is arr[l..m]
8          // Second subarray is arr[m+1..r]
9
10         int n1 = m - l + 1;
11         int n2 = r - m;
12
13         /* Create temp arrays and copy data into them */
14         int L[] = new int [n1];
15         int R[] = new int [n2];
16         for (int i=0; i<n1; ++i)
17             L[i] = arr[l + i];
18         for (int j=0; j<n2; ++j)
19             R[j] = arr[m + 1+ j];
20
21
22         /* Merge the temp arrays */
23
24
```

```
25          int i = 0, j = 0;  // indexes of first and second subarrays
26          int k = l;         // index of merged subarry array
27
28          while (i < n1 && j < n2) {
29              if (L[i] <= R[j]) {
30                  arr[k] = L[i];
31                  i++;
32              } else {
33                  arr[k] = R[j];
34                  j++;
35              }
36              k++;
37          }
38
39          /* Copy remaining elements of L[] if any */
40          while (i < n1) {
41              arr[k] = L[i];
42              i++;
43              k++;
44          }
45
46          /* Copy remaining elements of R[] if any */
47          while (j < n2) {
48              arr[k] = R[j];
49              j++;
50              k++;
51          }
52      }
```

```
54      // Main function that sorts arr[l..r] using
55      // merge()
56      void sort(int arr[], int l, int r)
57      {
58          if (l < r) {
59              // Find the middle point
60              int m = (l+r)/2;
61
62              // Sort first and second halves
63              sort(arr, l, m);
64              sort(arr , m+1, r);
65
66              // Merge the sorted halves
67              merge(arr, l, m, r);
68          }
69      }
```

Running time estimates:

- Home pc executes 10^8 comparisons/second.
- Supercomputer executes 10^{12} comparisons/second.

Insertion Sort (N^2)

computer	thousand	million	billion
home	instant	2.8 hours	317 years
super	instant	1 second	1.6 weeks

Mergesort (N log N)

thousand	million	billion
instant	1 sec	18 min
instant	instant	instant

Lesson 1. Good algorithms are better than supercomputers.

from Sedgewick & Wayne http://www.cs.princeton.edu/courses/archive/spr07/cos226/lectures/04MergeQuick.pdf

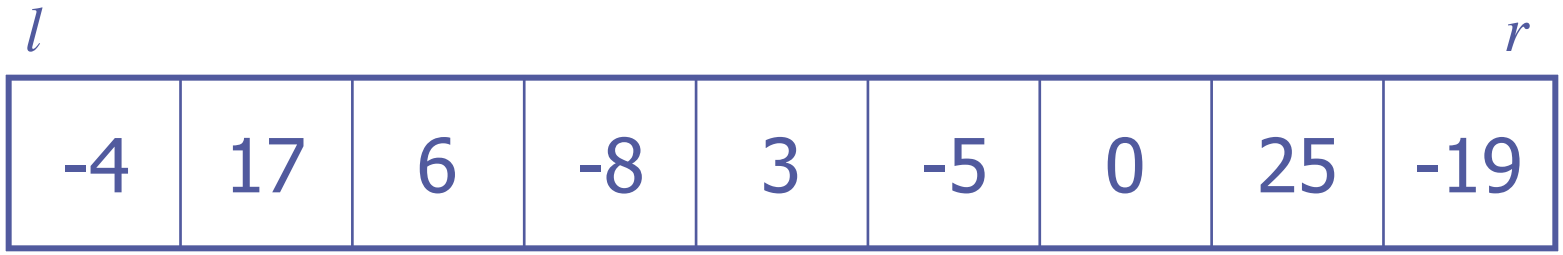
16

Problem

- Is Mergesort a stable sorting algorithm?
  - A yes
  - B no

Problem

- Design an algorithm to rearrange elements of a given array of n real numbers so that all its negative elements precede all its non-negative elements. Your algorithm should be both time-efficient and space-efficient.



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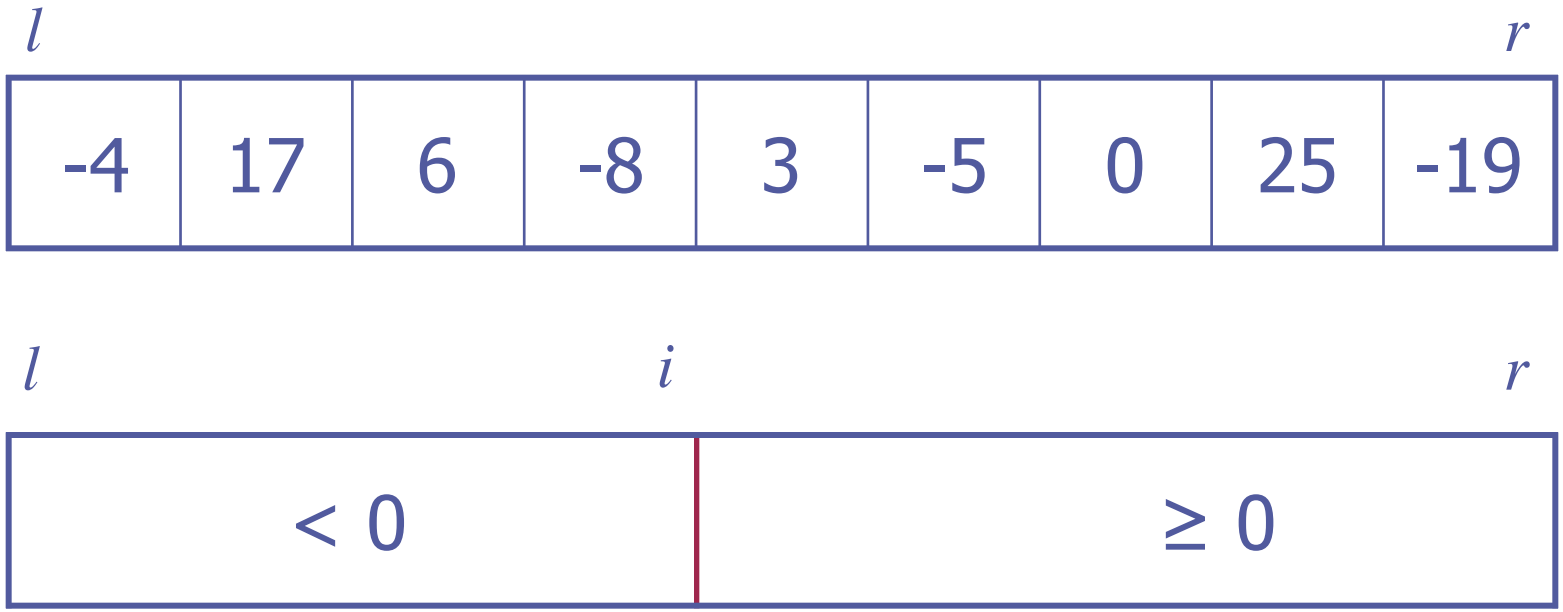

Partition based on Lomuto's idea

- Examine $A[i]$:

≥ 0: just increment i

< 0: make room in the segment $A[i..s]$ by
increment s;
swap $A[i]$ and $A[s]$;
increment i

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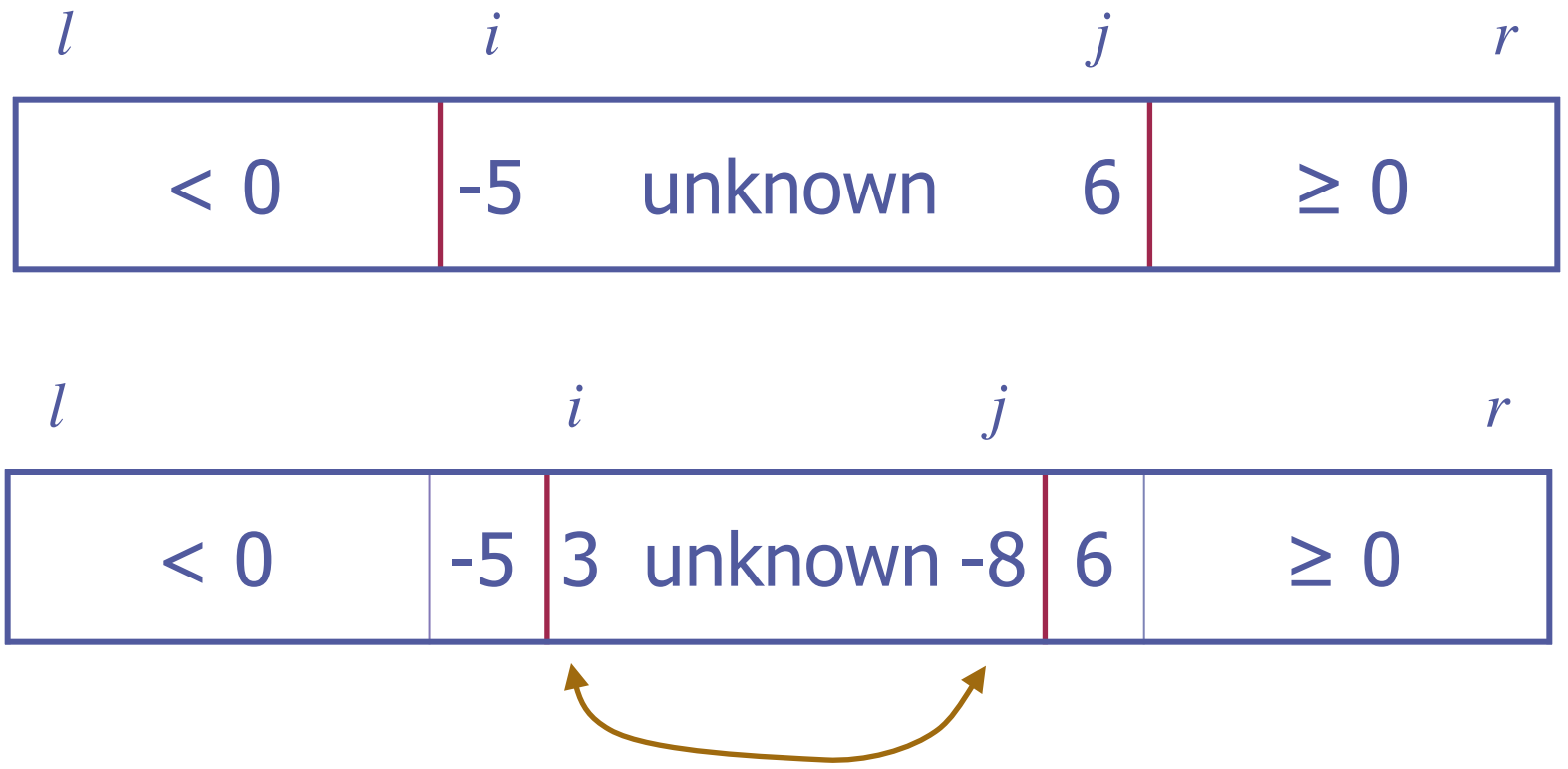

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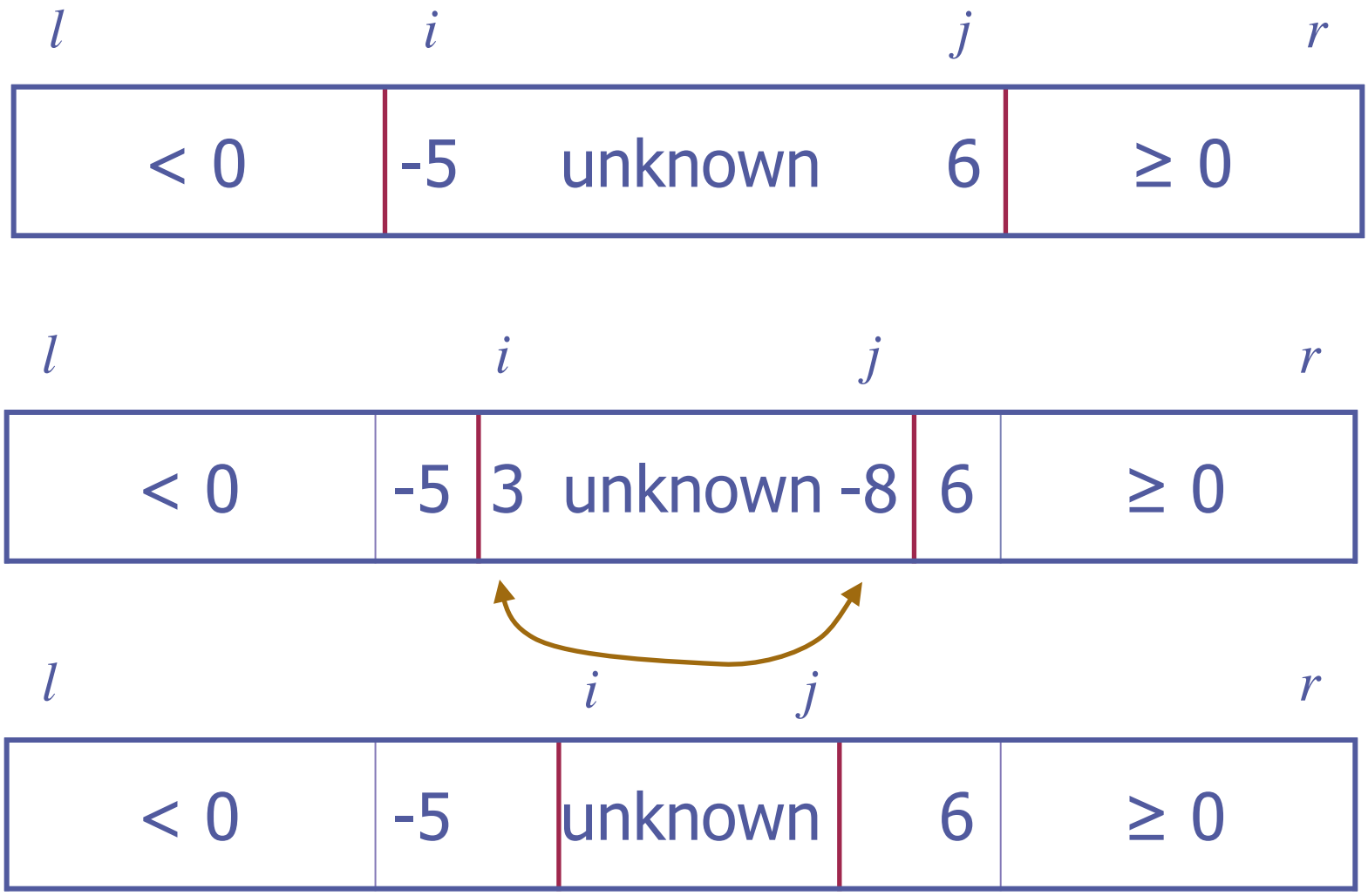

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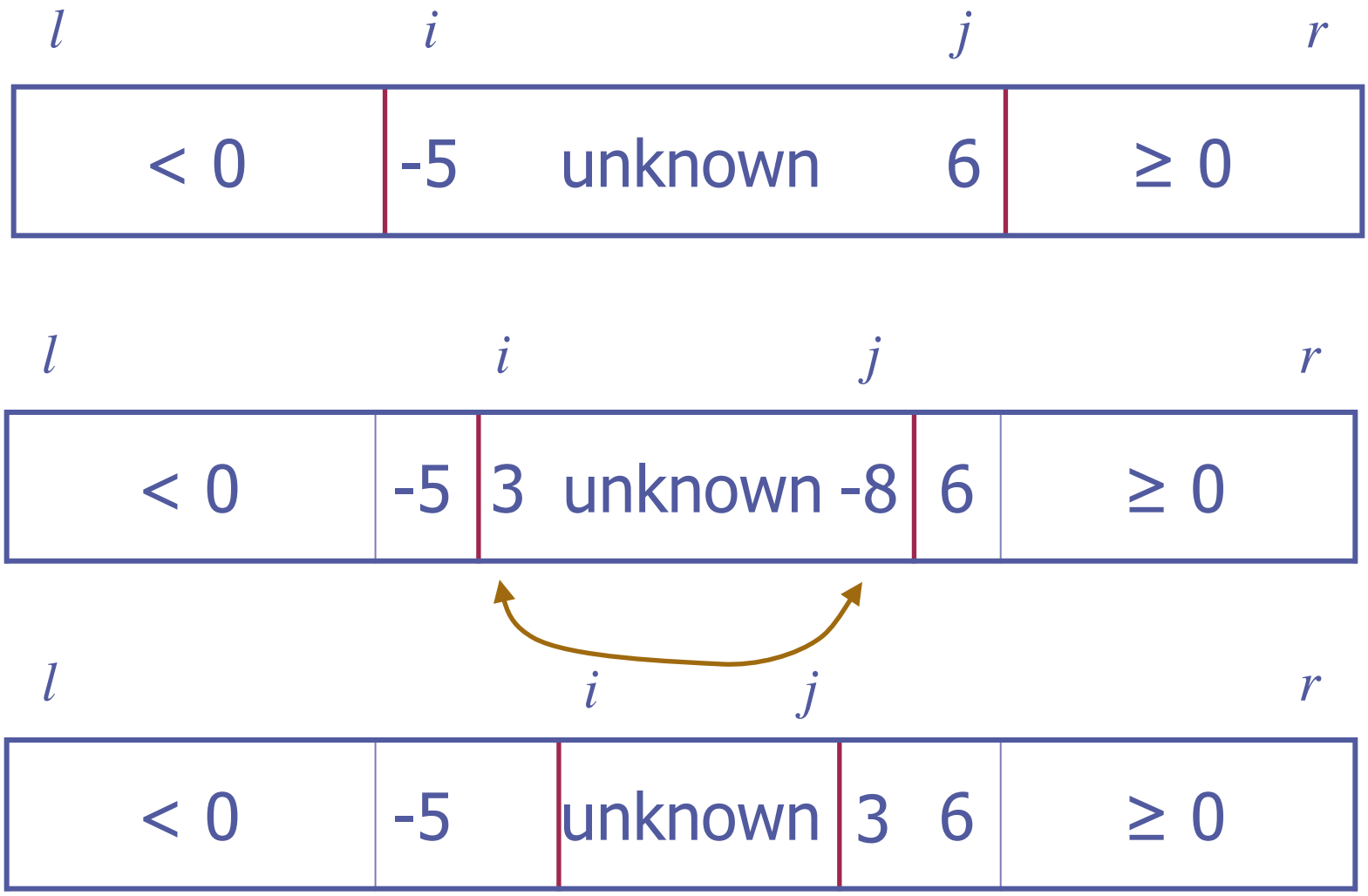

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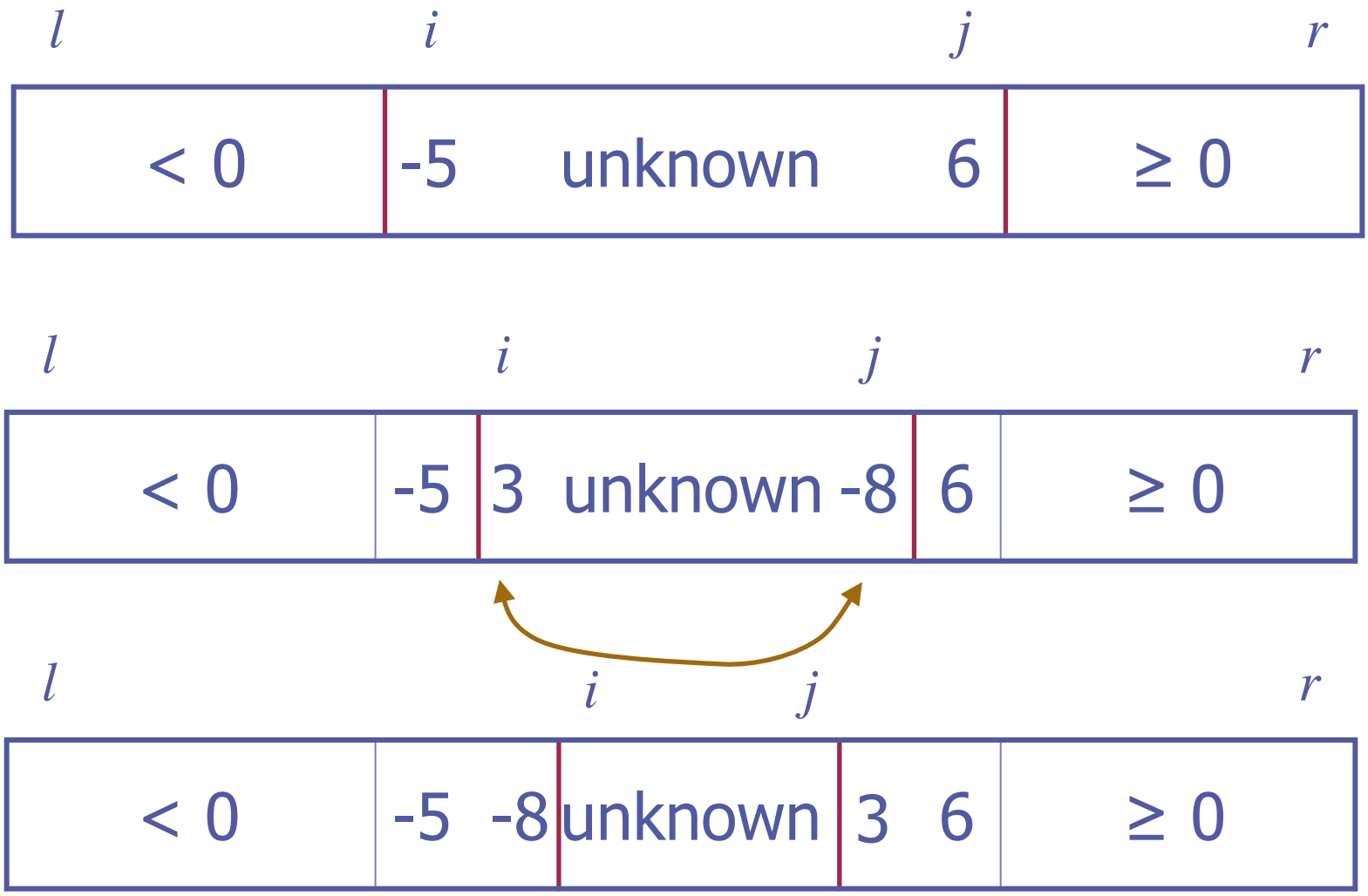
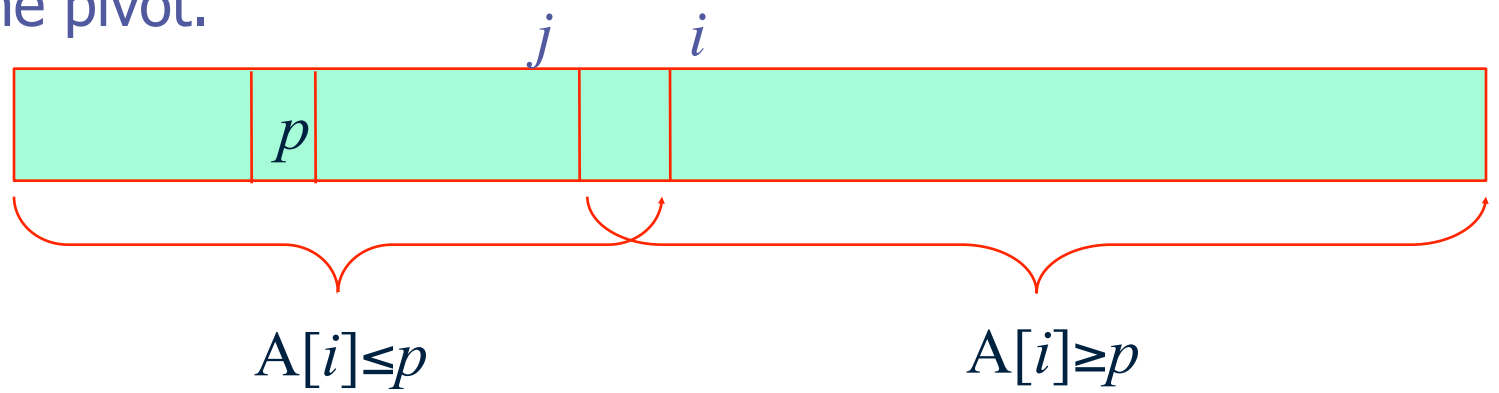


Quicksort

- Select a pivot (partitioning element) – a **random** element
- Rearrange the list so that all the elements in the first j positions are smaller than or equal to the pivot and all the elements in the last n-i positions are larger than or equal to the pivot.



- Exchange the pivot with the element in its partition closest to the center — the pivot is now in its final position
- Sort the two subarrays recursively

Hoare Partition Algorithm

```
method partition(A, lo, hi) {
        // A is an array; lo and hi are valid indices into A.
        // Returns a 2-element list [i, j] such that
        //     lo ≤ j < i ≤ hi,  provided lo < hi
        //     ∃ pivot st:
        //         A[r] ≤ pivot ∀ lo ≤ r ≤ j
        //         A[r] = pivot ∀ j < r < i
        //         A[r] ≥ pivot ∀ i ≤ r
    def pivotIndex = randomBetween(lo)and(hi)
    def pivot = A[pivotIndex]
    var i := lo−1
    var j := hi+1
    while {
        do { i := i + 1 } while { (i ≤ hi) && {A[i] ≤ pivot} }
        do { j := j − 1 } while { (j ≥ lo) && {A[j] ≥ pivot} }
        i < j
    } do { exchange(A, i, j) }
    if (i < pivotIndex) then {
                exchange(A, i, pivotIndex) ;  i := i + 1
    } elseif {j > pivotIndex} then {
                exchange(A, pivotIndex, j) ; j := j − 1
    }
    return [i, j]
}
```

Which Algorithmic Paradigm?

- **Partition**:

A. Brute force
B. Decrease by a constant
C. Decrease by a Variable Amount
D. Divide and Conquer

Which Algorithmic Paradigm?

- Median Finding using Partition:

A. Brute force
B. Decrease by a constant
C. Decrease by a Variable Amount
D. Divide and Conquer

Which Algorithmic Paradigm?

- Sorting using Partition (Quicksort):

A. Brute force
B. Decrease by a constant
C. Decrease by a Variable Amount
D. Divide and Conquer

Analysis of Quicksort

- Best case: split in the middle — $\Theta(n \log n)$
- Worst case: choose 1st element from sorted array! — $\Theta(n^2)$
- Average case: random arrays — $\Theta(n \log n)$

- Improvements:
  - better pivot selection: median-of-three partitioning
  - separate partition for keys equal to pivot
  - switch to insertion sort on small sub-problems
  - elimination of recursion
  - These combine to give 20–25% improvement

- Considered the method of choice for internal sorting of large files ($n \geq 10000$)

Dual-pivot Quicksort

Vladimir Yaroslavskiy | 11 Sep 12:35 2009

Replacement of Quicksort in java.util.Arrays with new Dual-Pivot Quicksort

```
Hello All,

I'd like to share with you new Dual-Pivot Quicksort which is
faster than the known implementations (theoretically and
experimental). I'd like to propose to replace the JDK's
Quicksort implementation by new one.
```

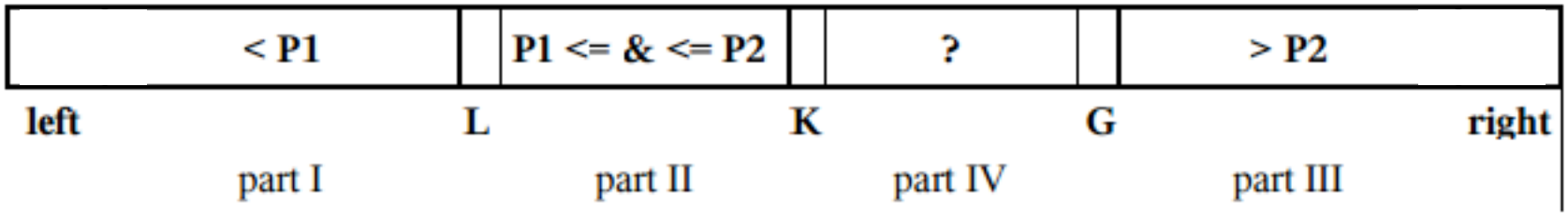
...

```
It is proved that for the Dual-Pivot Quicksort the average number of
comparisons is 2*n*ln(n), the average number of swaps is 0.8*n*ln(n),
whereas classical Quicksort algorithm has 2*n*ln(n) and 1*n*ln(n)
respectively.
```

- Read for yourself: https://web.archive.org/web/20150120151928/http://permalink.gmane.org/gmane.comp.java.openjdk.core-libs.devel/2628

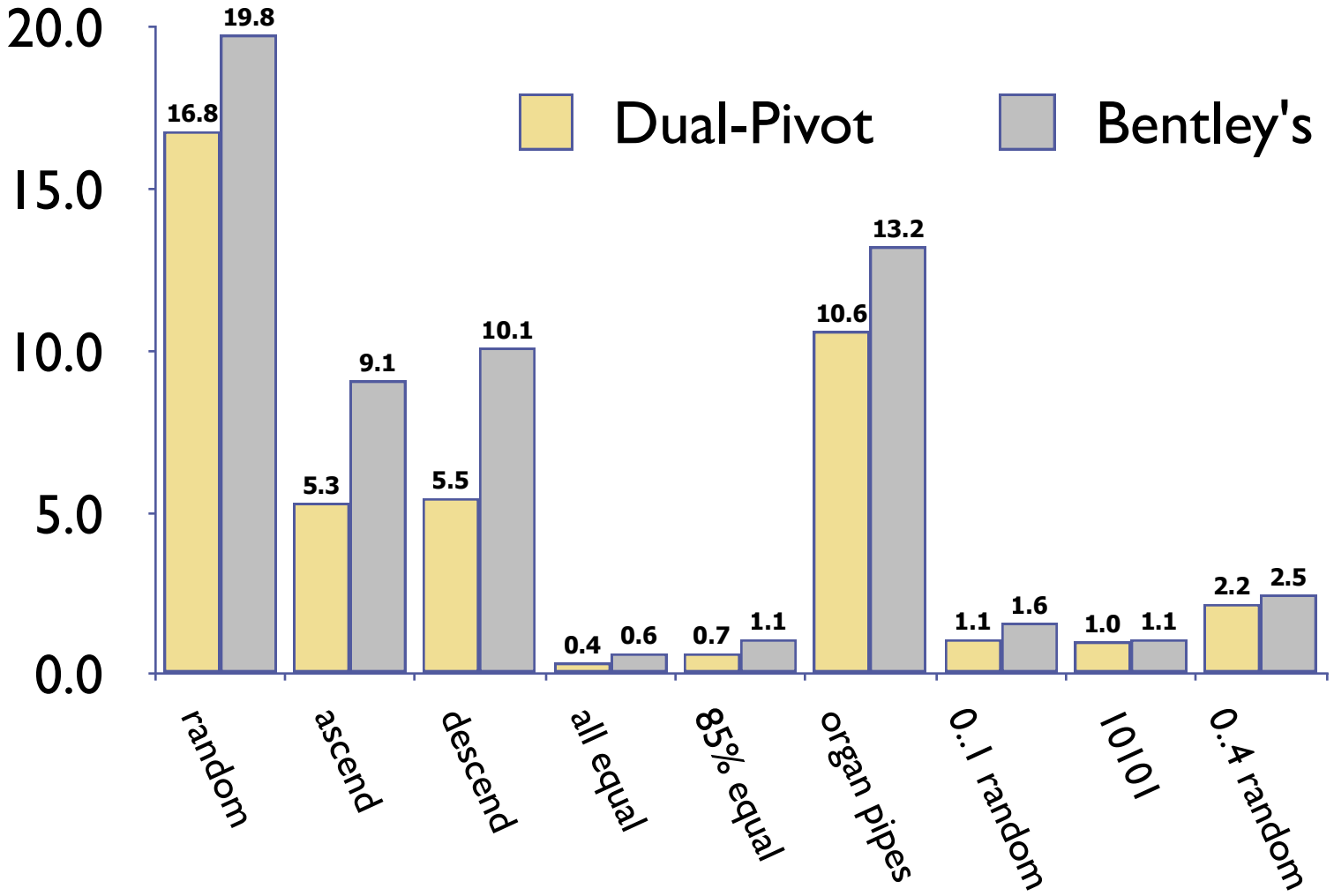
Basic idea

Figure 1.



- Uses <u>two</u> pivots, p_1 and p_2, such that $p_1 \leq p_2$

Performance Comparison



Jon Bentley (9th Sept 2009):

Vladimir, Josh,

I *finally* feel like I understand what is going on. Now that I (think that) I see it, it seems straightforward and obvious.

Tony Hoare developed Quicksort in the early 1960s. I was very proud to make minor contributions to a particularly clean (binary) quicksort in the mid 1980s, to a relatively straightforward, industrial-strength Quicksort with McIlroy in the early 1990s, and then to algorithms and data structures for strings with Sedgewick in the mid 1990s.

I think that Vladimir's contributions to Quicksort go way beyond anything that I've ever done, and rank up there with Hoare's original design and Sedgewick's analysis. I feel so privileged to play a very, very minor role in helping Vladimir with the most excellent work!

Problem:

- Is Quicksort stable?

A. Yes
B. No

Problem:

Should we stop scanning when we find an element = pivot?

- Levitin: **repeat** $i \leftarrow i + 1$ **until** $A[i] \geq pivot$
- Hoare: **if** $pivot < A[I]$ **then goto** $down$

✧

Problem:

Should we stop scanning when we find an element = pivot?

Levitin: **repeat** $i \leftarrow i + 1$ **until** $A[i] \geq pivot$

Hoare: **if** $pivot < A[I]$ **then goto** *down*

Levitin claims: *"Why is it worth stopping the scans after encountering an element equal to the pivot? Because doing this tends to yield more-even splits for arrays with a lot of duplicates, which makes the algorithm run faster."*

QUICKSORT IS OPTIMAL

Robert Sedgewick
Jon Bentley

- "Knuthfest", Stanford University, January, 2002.
  - http://www.sorting-algorithms.com/static/QuicksortIsOptimal.pdf
- Examines an interesting detail: How to handle keys equal to the pivot

Partitioning with equal keys

How to handle keys equal to the partitioning element?

METHOD A: Put equal keys all on one side?

```
4  4  4  4  4  4  4  4  4  4  4  4  4
4  4  4  4  4  4  4  4  4  4  4  4 (4)
```

NO: quadratic for n=1 (all keys equal)

METHOD B: Scan over equal keys? (linear for n=1)

```
1  4  1  1  4  4  4  1  4  1  1  4  4
1 (1) 1  1  4  4  4  1  4  1 (4) 4  4
```

NO: quadratic for n=2

METHOD C: Stop both pointers on equal keys?

```
4  9  4  4  1  4  4  4  9  4  4  1  4
1  4  4  4  1 (4) 4  4  9  4  9  4  4
```

YES: NlgN guarantee for small n, no overhead if no equal keys

Partitioning with equal keys

How to handle keys equal to the partitioning element?

METHOD C: Stop both pointers on equal keys?

YES: NlgN guarantee for small n, no overhead if no equal keys

METHOD D (3-way partitioning): Put all equal keys into position?

yes, BUT: early implementations cumbersome and/or expensive

Quicksort common wisdom (last millennium}

1. Method of choice in practice
   - tiny inner loop, with locality of reference
   - NlogN worst-case "guarantee" (randomized)
   - but use a radix sort for small number of key values
2. Equal keys can be handled (with care)
   - NlogN worst-case guarantee, using proper implementation
3. Three-way partitioning adds too much overhead
   - "Dutch National Flag" problem
4. Average case analysis with equal keys is intractable
   - keys equal to partitioning element end up in both subfiles

Changes in Quicksort common wisdom

1. Equal keys abound in practice.
   - never can anticipate how clients will use library
   - linear time required for huge files with few key values
2. 3-way partitioning is the method of choice.
   - greatly expands applicability, with little overhead
   - easy to adapt to multikey sort
   - no need for separate radix sort
3. Average case analysis already done!
   - Burge, 1975
   - Sedgewick, 1978
   - Allen, Munro, Melhorn, 1978

Bentley-McIlroy 3-way partitioning

Partitioning invariant

- move from left to find an element that is not less
- move from right to find an element that is not greater
- stop if pointers have crossed
- exchange
- if left element equal, exchange to left end
- if right element equal, exchange to right end

Swap equals to center after partition

KEY FEATURES

- always uses N-1 (three-way) compares
- no extra overhead if no equal keys
- only one "extra" exchange per equal key

Quicksort with 3-way partitioning

```
void quicksort(Item a[], int l, int r)
{ int i = l-1, j = r, p = l-1, q = r; Item v = a[r];
  if (r <= l) return;
  for (;;)
    {
      while (a[++i] < v) ;
      while (v < a[--j]) if (j == l) break;
      if (i >= j) break;
      exch(a[i], a[j]);
      if (a[i] == v) { p++; exch(a[p], a[i]); }
      if (v == a[j]) { q--; exch(a[j], a[q]); }
    }
  exch(a[i], a[r]); j = i-1; i = i+1;
  for (k = l; k < p; k++, j--) exch(a[k], a[j]);
  for (k = r-1; k > q; k--, i++) exch(a[i], a[k]);
  quicksort(a, l, j);
  quicksort(a, i, r);
}
```

Sorting analysis summary

Running time estimates:

- Home pc executes 10^8 comparisons/second.
- Supercomputer executes 10^{12} comparisons/second.

Insertion Sort (N^2)

computer	thousand	million	billion
home	instant	2.8 hours	317 years
super	instant	1 second	1.6 weeks

Mergesort (N log N)

thousand	million	billion
instant	1 sec	18 min
instant	instant	instant

Quicksort (N log N)

thousand	million	billion
instant	0.3 sec	6 min
instant	instant	instant

Lesson 1. Good algorithms are better than supercomputers.

Lesson 2. Great algorithms are better than good ones.

The Problem of the Dutch National Flag

- Levitin §5.2, Q9

The ***Dutch flag problem*** is to rearrange an array of characters R, W, and B (red, white, and blue are the colors of the Dutch national flag) so that all the R's come first, the W's come next, and the B's come last. Design a linear in-place algorithm for this problem.

Quickhull

- Divide-and-conquer algorithm for Convex Hull of a set of points P

  - Find two points A,B$\in$P that are both on the convex hull

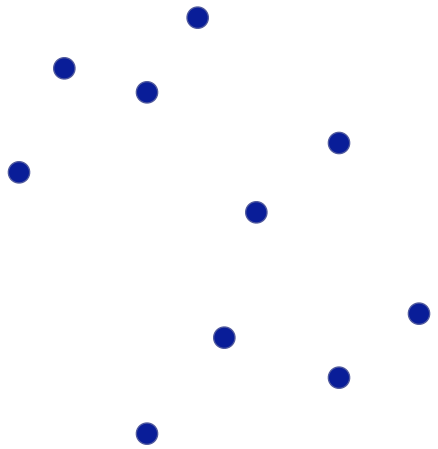
divide the set P into two subsets, P_l left and P_r right of chord AB.

Find point C in P_l furthest from AB.

  - discard the points inside $\triangle$ABC
  - recurse with chords AC and CB

    …

repeat with chord AB and P_r



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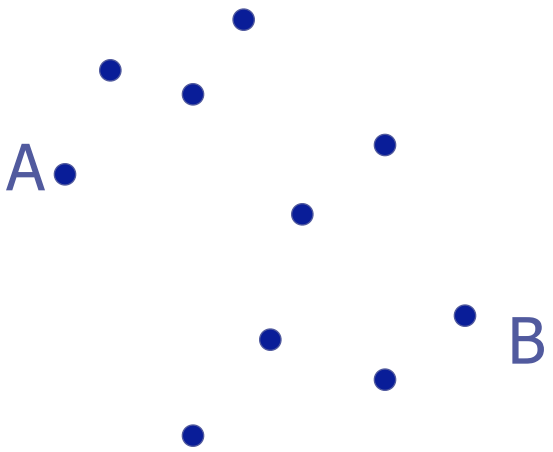

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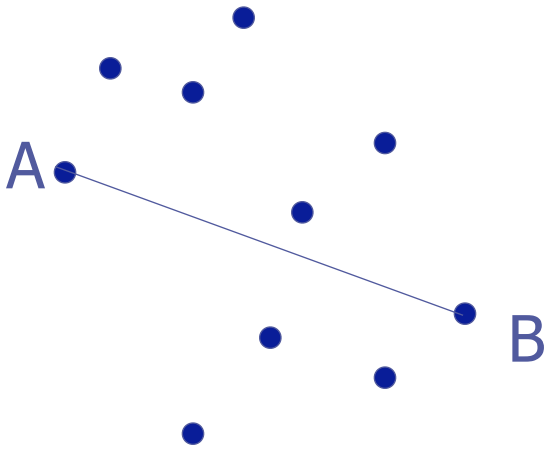

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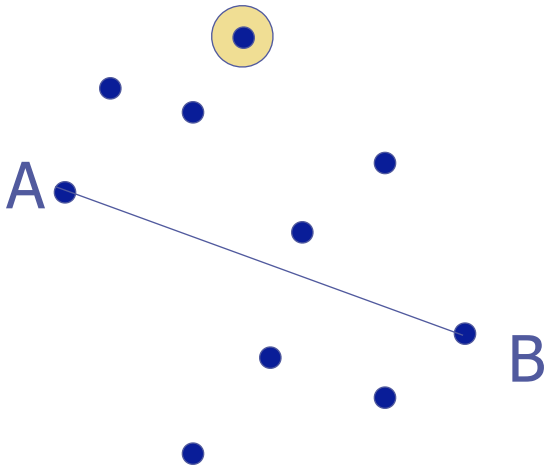

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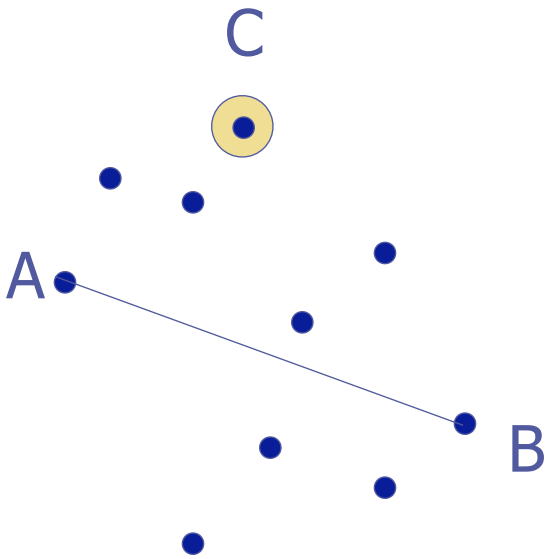

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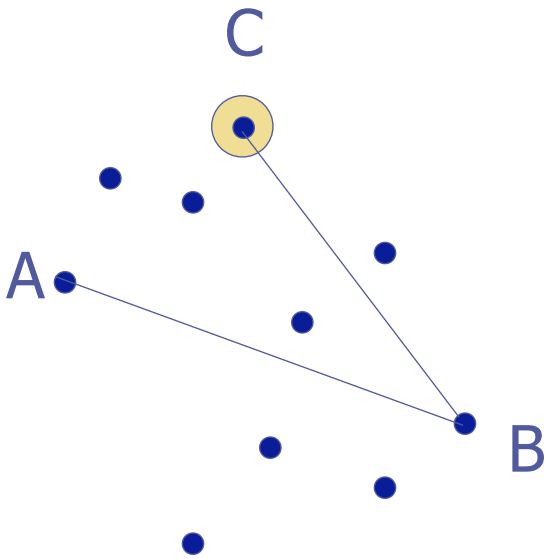

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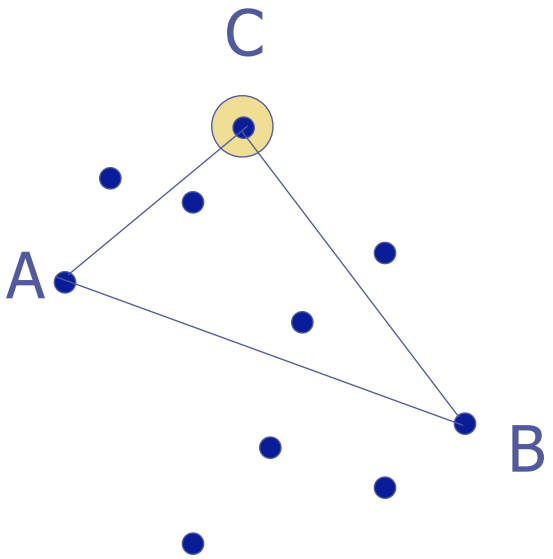

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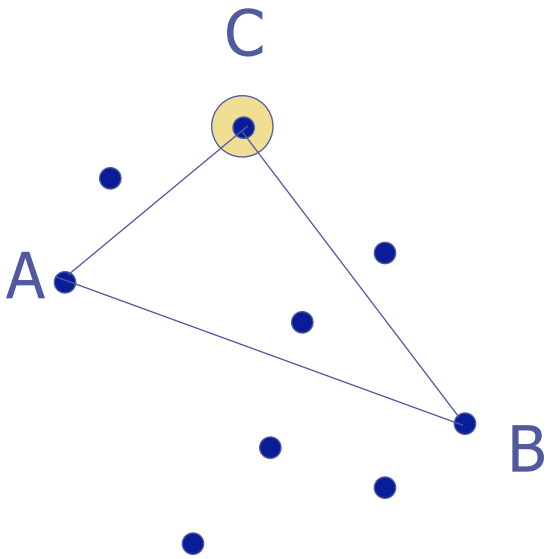

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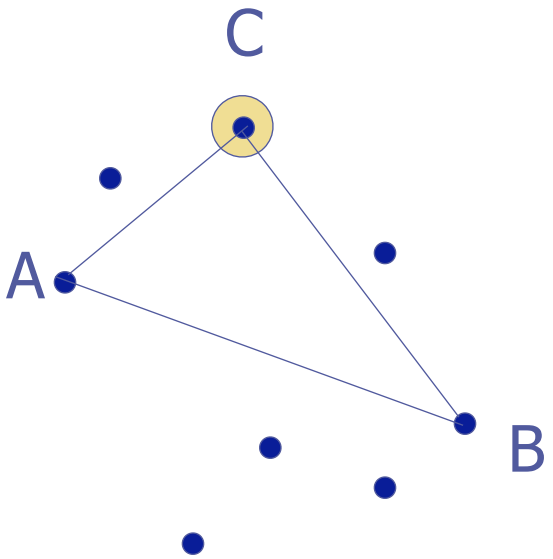

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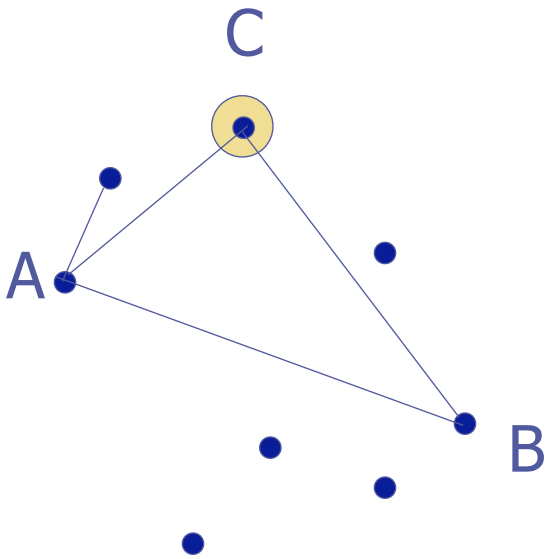

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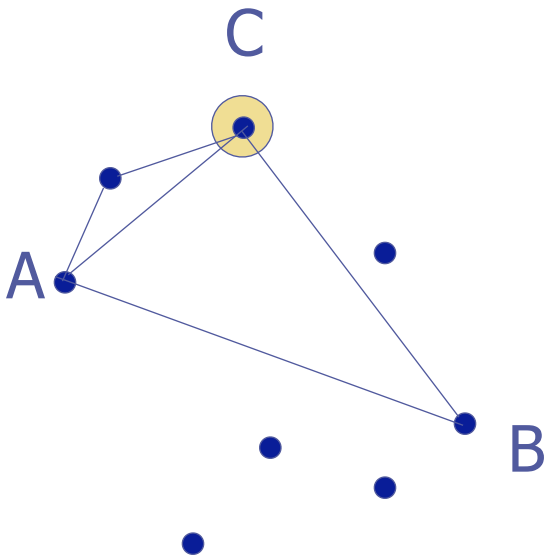
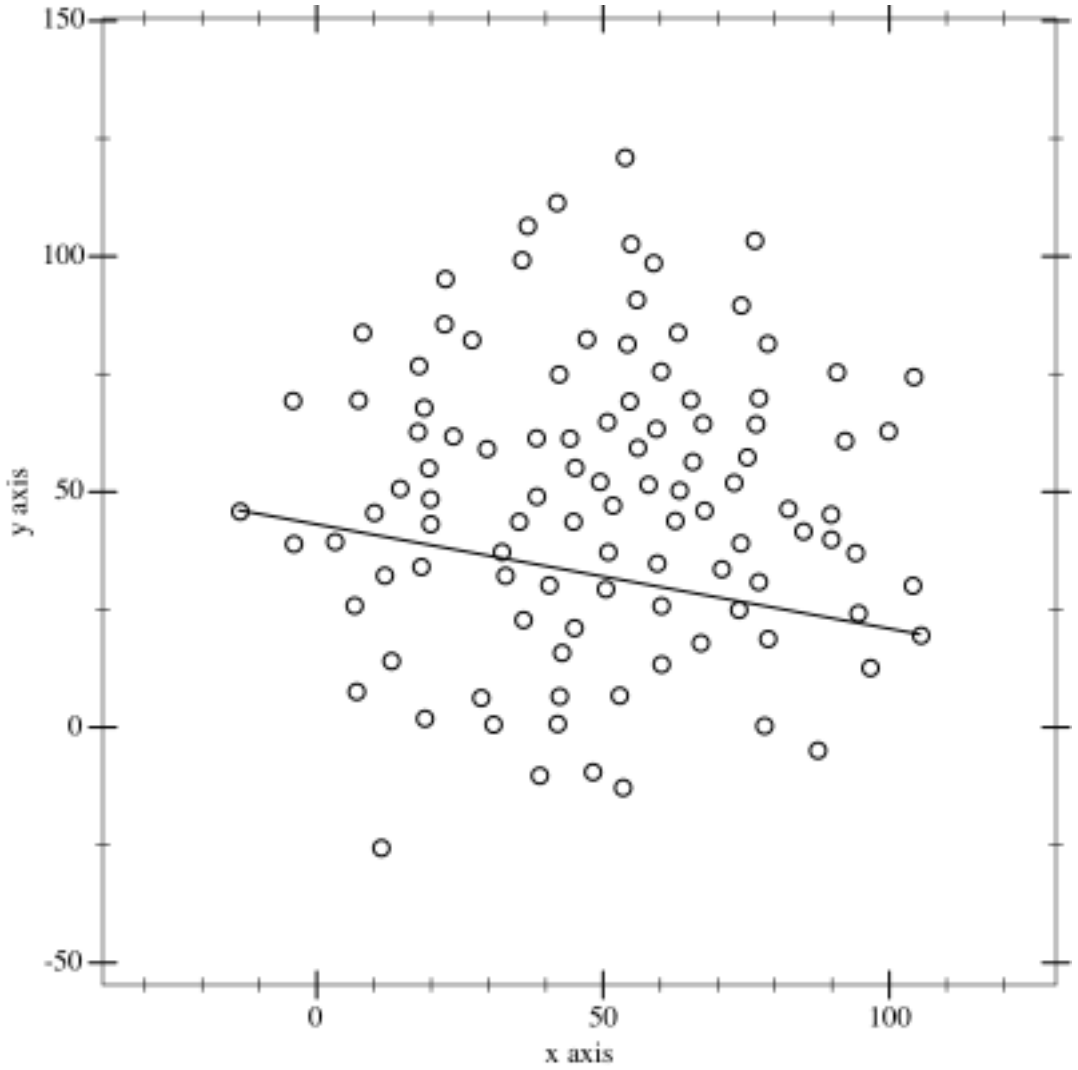
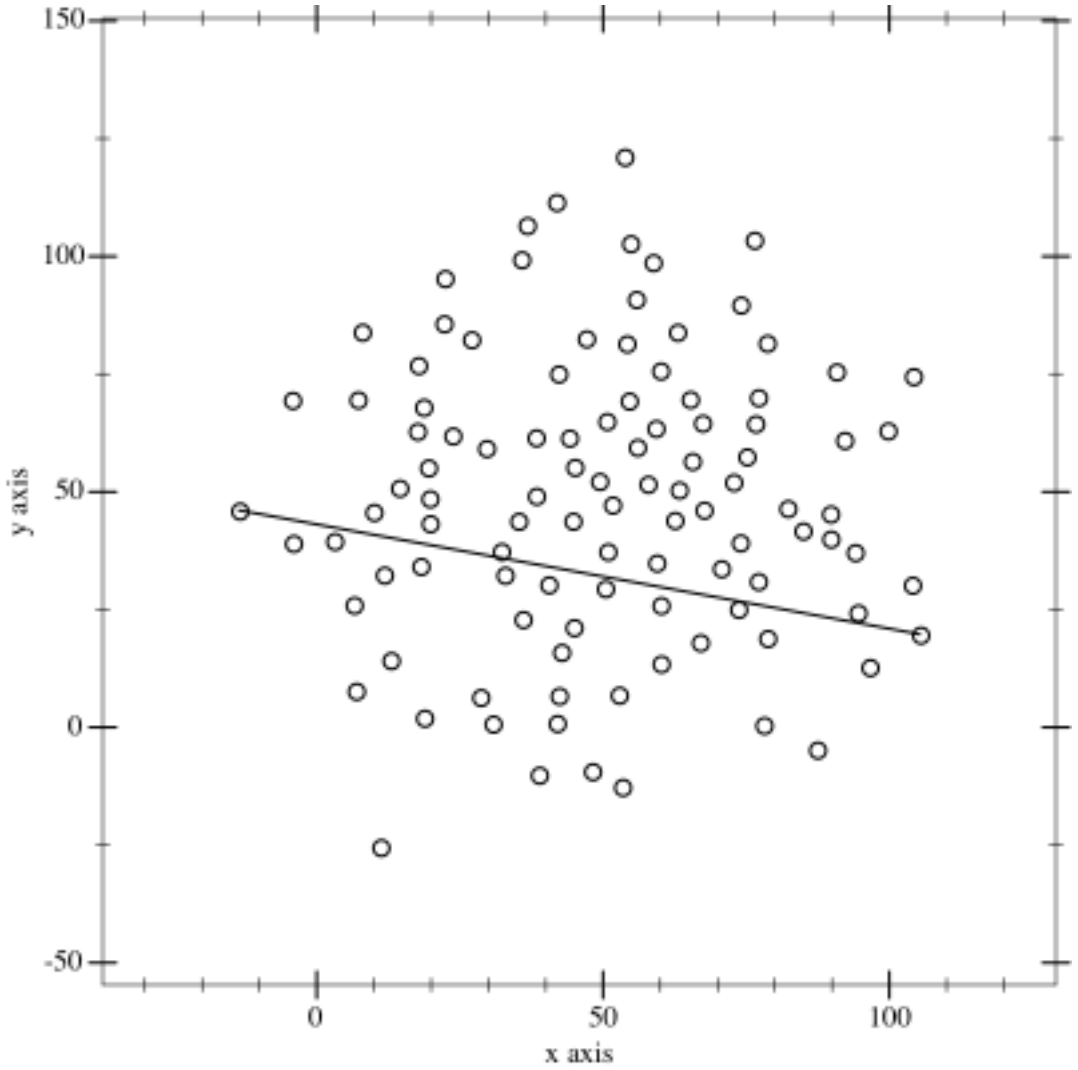


Animation:

- Quickhull at Princeton (needs Java)



Animation by Maonus - Own work, CC BY-SA 4.0, https://commons.wikimedia.org/w/index.php?curid=37505459



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