FIRST ORDER PREDICATE LOGIC

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AN OVERVIEW

FIRST-ORDER PREDICATE LOGIC

$$\forall q\, \exists p\, \forall x, y\, [p > q \wedge x, y > 1 \Rightarrow x \cdot y \neq p]$$

"There are infinitely many prime numbers."

$$\forall a, b, c, n\, [(a, b, c > 0 \wedge n > 2) \Rightarrow a^n + b^n \neq c^n]$$

"Fermat's Last Theorem: $a^n + b^n = c^n$ has no integer solutions for $n > 2$."

$$\forall q\, \exists p\, \forall x, y\, [p > q \wedge (x, y > 1 \Rightarrow (x \cdot y \neq p \wedge x \cdot y \neq p+2))]$$

"THE TWIN PRIME CONJECTURE: There are infinitely many prime pairs, e.g., 29, 31."

OVERVIEW OF LOGIC

FORMULAS

$$\forall x\, \exists y\, [\, x > y \Rightarrow y < x\,]$$

STRINGS WITH A CERTAIN SYNTAX.

UNIVERSE

A SET OF "OBJECTS".

RELATIONS

BETWEEN OBJECTS IN THE UNIVERSE

MODEL

CONNECTION BETWEEN SYMBOLS IN A FORMULA AND OBJECTS, RELATIONS IN THE UNIVERSE.

TRUTH

SOME FORMULAS ARE TRUE (WITHIN A GIVEN UNIVERSE)

TRUTH

SOME FORMULAS ARE TRUE.

HOW CAN WE **PROVE** THEY ARE TRUE?

AXIOMS
RULES OF INFERENCE
LOGICAL DEDUCTION

CAN WE AUTOMATE THIS PROCESS?

THE SET OF TRUE FORMULAS.

PROBLEM: IS A FORMULA TRUE?

IS THE SET OF TRUE FORMULAS DECIDABLE?

FORMULAS

ALPHABET

$$\Sigma = \{ \forall \; \exists \; (\;) \; \wedge \; \vee \; \Rightarrow \; \neg \; x \; R_1 \; R_2 \; R_3 \cdots \}$$

BOOLEAN OPERATIONS

$\wedge$ AND

$\vee$ OR

$\neg$ NOT

$\Rightarrow$ IMPLIES

QUANTIFIERS

$\forall$ UNIVERSAL ("for all")

$\exists$ EXISTENTIAL ("there exists")

VARIABLES

$x \quad y \quad z \quad \cdots$

$x_1 \quad x_2 \quad x_3 \quad \cdots$

$x \quad xx \quad xxx \quad xxxx \quad \cdots$

There is an infinite supply of variable names

RELATIONS

$R_1 \quad R_2 \quad R_3 \quad \cdots$

Often, we prefer to use symbols and syntax from the model; This makes the connection between relation symbols and relations in the universe transparent.

$R_2(x, y, z) \equiv x + y = z$

$+ \; - \; \times \; \div$

$\leq \; \geq \; = \; \neq$

etc...

SYNTAX OF FORMULAS

Each Relation symbol has an arity.
It must be used correctly.

$$R_1(x, y, z) \equiv x + y = z$$

$$\left.\begin{array}{l} R_1(x) \\ R_1),x((, \end{array}\right\} \text{Not okay}$$

A FORMULA IS...

- A RELATION (or an "ATOMIC FORMULA")
  $R(x, 4, 7)$
- $F_1 \wedge F_2$
- $F_1 \vee F_2$
- $F_1 \Rightarrow F_2$
- $\neg F_1$
- $\forall x\, [F_1]$
- $\exists x\, [F_2]$
- (F_1)

Where F_1 and F_2 are themselves formulas and x is any variable

Syntactic Variations:

$$\neg F_1 \equiv {\sim} F_1 \equiv \overline{F_1}$$

$$\forall x \bullet (\dots)$$

$$\exists x \bullet (\dots)$$

Parentheses used as necessary
Rules of precedence

CAN WE CHECK IF A STRING IS A LEGAL FORMULA?

(WFF = "Well Formed Formula")

Sure. Decidable. Simple Parsing problem.

Free Variables.

$$\forall x.\left[\, R_1(x, y) \Rightarrow R_2(z, x) \,\right]$$

Bound

FREE

DEFN

A "STATEMENT" is a formula with NO free variables.

All variables are quantified.

$\forall x \exists y \,.\, R(x, y)$ ← STATEMENT

$\forall x \,.\, R(x, y)$ ← NOT A "STATEMENT"

PREDICATE LOGIC: TRUTH, MEANING AND PROOF

ALGEBRAIC MANIPULATIONS

$$\sim \exists x . P \equiv \forall x . \sim P$$

$$\sim \forall x . P \equiv \exists x . \sim P$$

$$\sim (P \land Q) \equiv \sim P \lor \sim Q$$

$$P \Rightarrow Q \equiv \sim P \lor Q$$

etc...

THESE DO NOT CHANGE THE MEANING.

GIVEN A PARTICULAR MODEL...

THE FORMULA IS EITHER TRUE OR FALSE...

THESE MANIPULATIONS DO NOT ALTER THE TRUTH OF THE FORMULA.

(Whatever the model may be.)

PRENEX FORM

ALL QUANTIFIERS ARE AT THE FRONT.

$$\forall x\, \exists y\, \forall z.\, (\ldots \wedge \ldots \vee \ldots \Rightarrow \ldots x \ldots y \ldots z \ldots)$$

- We can use the algebraic manipulations to put any formula into PRENEX form.
- So, without loss of generality... Assume henceforth that all formulas are in PRENEX form.

WHAT DOES A STATEMENT "MEAN"?

How do we **INTERPRET** it?

Is it TRUE or FALSE?

WE NEED...

- A <u>UNIVERSE</u> eg: $\mathbb{N} = \{0, 1, 2, \cdots\}$
- INTERPRETATIONS FOR THE RELATION SYMBOLS.

$$R_1(x, y) \equiv x \leq y$$

$$R_2(x, y, z) \equiv x + y = z$$

Symbols in the alphabet

Relations with meaning in the "UNIVERSE"

DEFN: A "MODEL" consists of $(\mathbb{U}, P_1, R_2, \cdots P_K)$

$$\begin{array}{ccc} P_1 & P_2 & P_K \\ ||| & ||| & ||| \\ R_1 & R_2 \cdots & R_K \end{array}$$

Universe

Relations in the Universe, e.g. $\leq$

Symbols in the formula

TRUTH

IS A GIVEN STATEMENT TRUE?

Must specify which Model!

YES (eg: Statement of Infinite # of Primes)

NO

MAYBE/UNKNOWN (eg: Twin Prime Conjecture)

EXAMPLE

$$\forall x,y,z.\left(R(x,y) \land R(y,z) \Rightarrow R(x,z)\right)$$

TRUE

INTERPRETATION #1: $R \equiv <$ $\quad \mathbb{U} \equiv \mathbb{N}$

$$\forall x,y,z\left(x<y \land y<z \Rightarrow x<z\right)$$

FALSE

INTERPRETATION #2: $R \equiv \text{SUCC}$ $\quad \mathbb{U} \equiv \mathbb{N}$

$$\forall x,y,z\left(x+1=y \land y+1=z \Rightarrow x+1=z\right)$$

$\boxed{\text{TAUTOLOGY}}$

Always true

... IN ANY MODEL.

$$\forall x. \left(R_1(x) \wedge R_2(x) \Rightarrow R_1(x) \right)$$

PREDICATE LOGIC

TRUE STATEMENTS AND PROVABLE STATEMENTS

TRADITIONAL APPROACH TO MATH PROOFS

Axioms

A given set of statements
assumed to be true
without proof.

Rules of inference/deduction

A set of rules for transforming
one statement into another.

Preserve truth.

Each rule is an algebraic
(i.e., Algorithmic, Computable) procedure.

Proof

A sequence of statements from
axioms to theorems, using only
the rules of inference.

Theorems

- Proof is found (creative search)
- Proof is verified (computer?) to make sure it is legit.

Interpretation?

"Intuition" about the model may guide the search for proofs.

Or: The search can be conducted <u>WITHOUT</u> <u>ANY</u> <u>UNDERSTANDING</u>.

The symbols in any statement are just that: <u>MEANINGLESS</u> <u>SYMBOLS</u>.

Validity of this approach?

How else can one define "TRUTH"?

CONSIDER THE TWIN PRIME CONJECTURE.

x and $x+2$ are both prime.
There are an infinite #
of these twin primes.

$\boxed{\text{Either it is TRUE or it is NOT TRUE.}}$

Let's fix the Universe and
interpretation of symbols.
(the Model).

EXAMPLE

UNIVERSE $= \mathbb{N} = \{0, 1, 2, \cdots\}$

Relations: $+ \quad - \quad * \quad < \quad = \quad \geq \quad \ldots$

[This is "Number Theory"]

Let's ask about the
Set of true statements
(It's a language, after all!)

DEFN

Given a particular model, $\mathcal{M}$,
[eg: Numbers, +, *, ...]
the set of true statements
is called "The THEORY OF $\mathcal{M}$."

$$\text{Th}(\mathcal{M})$$

[EXAMPLE: Number Theory

$$\text{Th}(\mathbb{N}, +, *)$$

is the set of all True statements you can make using
+ * = etc.

NOTE: THIS IS <u>NOT</u> NECESSARILY THE SET OF PROVABLE STATEMENTS!
(eg With Axioms, Rules of inference).

THEOREM

If we limit ourselves to statements we can make only using + [that is, without *] Then, the set of true statements is decidable.

$\mathrm{Th}(\mathbb{N}, +)$ is DECIDABLE

Given a statement, there is a procedure to tell whether it is true or not.

$$\forall x, y, z, a, b, c \ ((x+y=z \wedge x+x=a \wedge y+y=b \wedge z+z=c) \Rightarrow a+b=c)$$

$$\begin{array}{c} x+y=z \\ x+y=z \\ \hline a+b=c \end{array}$$

NUMBER THEORY AND GÖDEL'S INCOMPLETENESS THEOREM

$Th(\mathbb{N}, +, *)$

THEOREM

Number Theory is Undecidable.

Considering the Universe to be

$$U = \mathbb{N} = \{0, 1, 2, \dots\}$$

and limiting ourselves to simple operations like $+$ and $*$, the set of true statements is undecidable.

PROOF IDEA

REDUCE A_{TM} TO THE PROBLEM OF DECIDING $Th(\mathbb{N}, +, *)$.

Kurt Gödel's Incompleteness Theorem

Formal Proof

A sequence of statements.
- Starting with axioms.
- Using precise rules of inference.
- Ending with theorem.

Proof of ϕ: ← A statement (not {})

$$\pi = (S_1, S_2, \ldots, S_L = \phi)$$

Assumption: Correctness

Proofs can be checked/verified.

$$\boxed{\{\phi, \pi \mid \pi \text{ is a proof of } \phi\} \text{ is decidable}}$$

Assumption: Soundness (consistency)

If a proof exists,
then the statement is true.

THEOREM

The set of provable statements in Number Theory

$$Th(\mathbb{N}, +, *)$$

is Turing Recognizable.

So we can enumerate all the PROVABLE statements

PROOF

- Finite Set of Axioms.
- Finite Set of Rules-of-inference.
- Each proof is finite in length.
- Each formula is finite in length.
- Just start listing them all out.

  → Enumerate all Proofs.

THEOREM

Some statement ψ

is true but

is not provable!

[Some statement in $Th(\mathbb{N}, +, *)$

has no proof.]

"Simple arithmetic may contains

truths which are inaccessible."

"Number Theory is way-deep."

PROOF (By Contradiction)

Assume all true statements are provable.

Here is an algorithm to DECIDE the truth of a statement:

Look for a proof of ϕ.

Look for a proof of $\neg\phi$.

Do these searches simultaneously, in parallel.

One or the other will be true.

Eventually we'll find a proof of either ϕ or $\neg\phi$.

But we know that Number Theory is Undecidable!

(We can't decide the TRUTH of statements.)

CONTRADICTION!

A STATEMENT THAT IS TRUE BUT NOT PROVABLE...

$\boxed{\text{IDEA:}}$

"This sentence is not provable".

$\psi_{\text{UNPROVABLE}}$ = an encoding of this sentence into a statement in Number Theory.

Is it true?

Find a proof $\Rightarrow$ contradiction!

Therefore, it must not be provable.

Therefore it is true!

PROBLEM:

The statement contains "this sentence".

SOLUTION:

The RECURSION THEOREM!