CS340: Discrete Structures **Exam #1**

Name ANSWER KEY

Open book, open notes, calculator use okay.

~~102~~ total

1. (8 points) Give a proof of the following statement about integers. Use only the definitions of odd and even, together with algebra. Do not use any other known information about odd and even numbers. Use complete statements and make your chain of reasoning clear.

If $5x + y$ is odd and x is even, then y is odd.

Since $5x+y$ is odd, $5x+y = 2j+1$ for some $j \in \mathbb{Z}$.
Since x is even, $x = 2k$ for some $k \in \mathbb{Z}$
$5x+y = 5(2k)+y = 2j+1$
$10k+y = 2j+1$
$y = 2j-10k+1$
$y = 2(j-5k)+1$
Since $j-5k \in \mathbb{Z}$, y is in the form $2n+1$.
Therefore y is odd. QED.

2. (12 points) Evaluate each expression.

a. power($\{a, b\}$) = $\{\emptyset, \{a\}, \{b\}, \{a, b\}\}$

b. $\{a, b, c\} - (\{a, b\} \cap \{b, c, d\})$ = $\{a, c\}$

c. $\{a, b\} \times \{7, 8, 9\}$ = $\{(a,7), (a,8), (a,9), (b,7), (b,8), (b,9)\}$

d. floor($\log_2(34)$) = $\log_2(32) < \log_2(34) < \log(64)$; $5 < \log_2(34) < 6$; 5

For the following expressions, let $f : \mathbf{N}_7 \to \mathbf{N}_7$ be defined by $f(x) = (2x+1) \bmod 7$.

x	$f(x)$
0	1
1	3
2	5
3	0
4	2
5	4
6	6

e. $f^{-1}(\{0, 1, 2\})$ = $\{0, 3, 4\}$

f. range(f) = $\{0, 1, \ldots, 6\} = \mathbb{N}_7$

3. (6 points) Let $A = \{3n + 6 \mid n \in \mathbf{N}\}$ and let $B = \{3k + 3 \mid k \in \mathbf{N}\}$. Prove that $A \subseteq B$.

Assume $x \in A$; want to show $x \in B$.
$x = 3n+6$ for some $n \in \mathbb{N}$.
$x = 3n+3+3$.
$x = 3(n+1)+3$.
Since $n+1 \in \mathbb{N}$, x is in the form $3k+3$ for some $k \in \mathbb{N}$
Therefore $x \in B$. QED

$4^5 = 2^{10} = 1024$

4. (5 points) Find an expression for the number of strings over the alphabet $\{a, b, c, d\}$ that have length 6 and such that the first letter in each string is either a or b, and in which each string contains at least one c.

$3^5 = 9 \cdot 9 \cdot 3 = 81 \cdot$
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Let U = strings of length 6 starting with a or b.
$|U| = 2 \cdot 4^5 = 2048$
Let S = strings that start with a or b and contain only a, b, or d
$|S| = 2 \cdot 3^5 = 486$
The set we want is $U - S$. Since $S \subseteq U$, $|U - S| = |U| - |S| = 2 \cdot 4^5 - 2 \cdot 3^5 =$
$2{,}048 - 486 = 1{,}562$

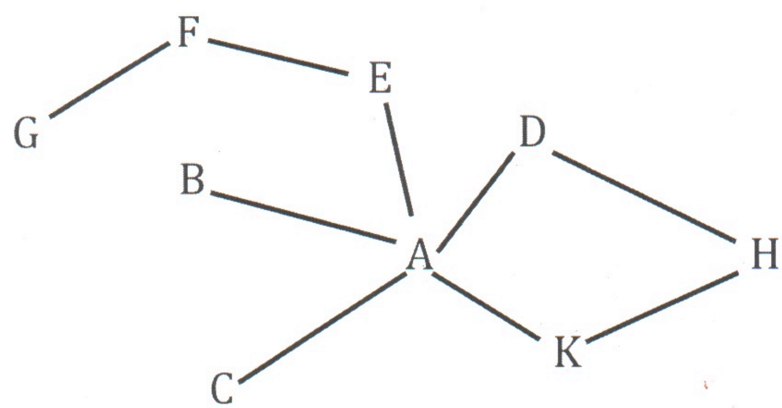
5. (5 points) Solve the following language equation for L.

$$\{\Lambda, b, ab\}L = \{\Lambda, a, b, ba, ab, aba\}.$$

$L = \{\Lambda, a\}$

Checking:
$\{\Lambda, b, ab\}\{\Lambda, a\} = \{\Lambda\Lambda, \Lambda a, b\Lambda, ba, ab\Lambda, aba\}$

6. (6 points) Given the following graph.



a. Write down the vertices of the graph in the order that they are visited by a breadth-first search of the graph that starts at vertex **A**.

A {B C E D K} {H F} G

EXAMPLE:
A C B E K D F H G

b. Write down the vertices of the graph in the order that they are visited by a depth-first search of the graph that starts at vertex **A**.

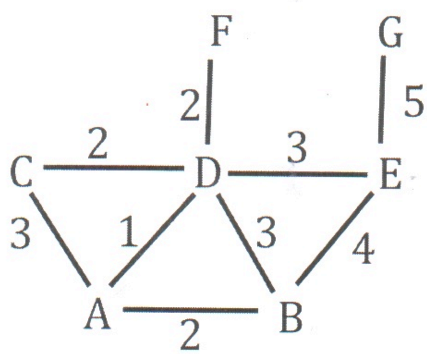
GENERAL FORM:

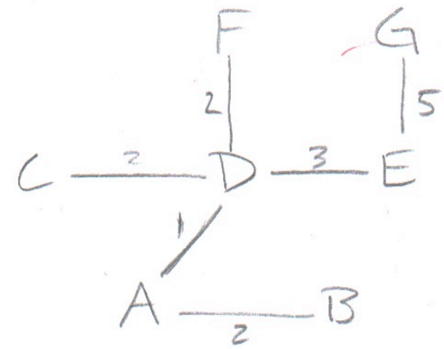
A {EFG, B, C, DHK / KHD}

EXAMPLE ANSWER:

A B C EFG KHD

7. (6 points) Draw ~~a~~ the minimal spanning tree for the following weighted graph:





8. (8 points) Let **Q+** denote the positive rational numbers. Let f : **Q+** → **Q+** be the function defined by $f(x) = x/(x + 3)$.

a. Show that f is injective.

f is injective if $f(x) = f(y)$ implies $x = y$. Assume $f(x) = f(y)$

$x/x+3 = y/y+3$

$y(x+3) = x(y+3)$

$xy + 3y = xy + 3x$

$3y = 3x$

$y = x$

QED.

b. Show that f is NOT surjective.

Assume f is surjective.

Therefore $f(x) = y$ for every $y \in Q^+$.

Consider $y = 1$.

$f(x) = 1$.

$x/x+3 = 1$.

$x = x + 3$.

$0 = 3$.

Contradiction!

Therefore f is not surjective. QED.

9. (6 points) Let f : $\mathbf{N}_8 \to \mathbf{N}_8$ be defined by $f(x) = (3x + 1) \bmod 8$. Find a formula for f^{-1}, the inverse of f.

x	$f(x)$
0	1
1	4
2	7
3	2
4	5
5	0
6	3
7	6

Checking → $f^{-1}(x)$

See 2.6 on page 109

$f(x) = (\underset{a}{3}x + \underset{b}{1}) \bmod \underset{n}{8}$

Since $\gcd(a, n) = \gcd(3, 8) = 1$, f is bijective & f^{-1} exists.

$f^{-1}(x) = (kx + c) \bmod 8$

$f(c) = 0 \Rightarrow c = \{\ldots, -11, -3, 5, 13, 21, \ldots\}$

$1 = ak + n \cdot m$ for some m

$1 = 3(k) + 8(m)$

$k = 11 \quad 3 \quad -5 \quad -13 \ldots$

$m = -4 \quad -1 \quad 2 \quad 5 \ldots$

EXAMPLE ANSWER

$f^{-1}(x) = (kx + c) \bmod 8$

$= (3x + 5) \bmod 8$

4 4 6 2 0 1 1

10. (6 points) Let

$S = \{$ June, July, August, September, October, November, December $\}$

and suppose that $h: S \rightarrow \mathbf{N}_7$ is the hash function defined as follows, where $|x|$ denotes the length of string x.

$$h(x) = |x| \bmod 7.$$

Use h to place each month of S into the following hash table starting with June, and proceeding in order (June, July, … December}. Resolve collisions by linear probing with a gap of 2.

0	October
1	August
2	September
3	November
4	June
5	December
6	July

11. (2 points) What is the cardinality of $\{2, 5, 8, \ldots 35, 38\}$?

$S = \{3K-1 \mid \text{for } K \in \{1, 2, \ldots, 13\}\}$. So $|S| = 13$

12. (4 points) Show that the set of negative integers is countable by establishing a bijection between the set an **N**.

$x = \mathbb{N} = \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad \ldots$

$f(x) = \text{Neg} = \quad -1 \quad -2 \quad -3 \quad -4 \quad -5 \quad \ldots$

Let $f(x) = -(x+1)$.

f is injective since $f(x) = f(y)$ implies $-(x+1) = -(y+1)$ implies $-x = -y = x = y$ for $x, y \in \mathbb{N}$

f is surjective since each $y \in \text{Neg}$ can be expressed as $f(x)$. $f(x) = y$ for $y < 0$

$-(x+1) = y < 0$

$-x - 1 < 0$

$-x < +1$

$x > -1$ implies $x \in \mathbb{N}$.

Therefore f is bijective. Since $\mathbb{N}$ is countable so is Neg. QED.

Oops! This should have been $\mathbb{N}$.

13. (12 points) Write out an inductive definition for each set.

a. $S = \{x \mid x \in \mathbf{Z} \text{ and } x \bmod 4 = 0\}$. $= \{0, 4, -4, 8, -8, 12, -12, \ldots\}$

This covers $-4, -8, -12, \ldots$

BASIS: $0 \in S$

INDUCTION: If $x \in S$ then $x+4 \in S$ and $x-4 \in S$.

b. $S = \{x \mid x \in \text{Lists}(\{0, 1\}) \text{ and } x \text{ has even length}\}$.

BASIS: $\langle\rangle \in S$

INDUCTION: If $x \in S$ then so is
$0 :: 0 :: x$
$0 :: 1 :: x$
$1 :: 0 :: x$
$1 :: 1 :: x$

c. $S = \mathbf{N} \times \{a, b\}^*$.

BASIS: $(0, \Lambda) \in S$

INDUCTION: If $(n, x) \in S$ then so are $(n+1, x)$, (n, ax), and (n, bx)

14. (16 points) Write a recursive definition for each of the following functions. Write your definitions in either if-then-else form or as pattern-matching equations.

a. $f : \mathbf{N} \to \mathbf{N}$ defined by $f(n) = 0 + 4 + 8 + \ldots + 4n$.

$f(0) = 0$
$f(n+1) = f(n) + 4(n+1)$

If $n = 0$ then 0
else $f(n-1) + 4 \cdot n$

b. $g : \text{Lists}(\mathbf{N}) \to \mathbf{N}$ defined by $g(\langle x_1, \ldots, x_n\rangle) = x_1 + \ldots + x_n$.

$g(\langle\rangle) = 0$
$g(x :: L) = x + g(L)$

If $L = \langle\rangle$ then 0
else $\text{head}(L) + g(\text{tail}(L))$

c. $h : \text{Lists}(A) \to \text{Lists}(A \times A)$ defined by

$$h(\langle x1, \ldots, xn\rangle) = \langle(x_1, x_1), \ldots, (x_n, x_n)\rangle.$$

$h(\langle\rangle) = \langle\rangle$
$h(x :: L) = (x, x) :: h(L)$

If $L = \langle\rangle$ then $\langle\rangle$
else $\text{cons}((\text{head}(L), \text{head}(L)), h(\text{tail}(L)))$

d. leaves: $\text{BinaryTrees}(\mathbf{N}) \to \mathbf{N}$ where leaves(T) is the number of leaves in the binary tree T.

$\text{leaves}(\langle\rangle) = 0$
$\text{leaves}(\langle\langle\rangle, a, \langle\rangle\rangle) = 1$
$\text{leaves}(L, a, R) = \text{leaves}(L) + \text{leaves}(R)$

If $T = \langle\rangle$ then
0
elseIf $T = \langle\langle\rangle, a, \langle\rangle\rangle$ then

or: $\text{left}(T) = \text{right}(T) = \langle\rangle$

1
else
$\text{leaves}(\text{left}(T)) + \text{leaves}(\text{right}(T))$.