Context Free Grammars

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Formal Languages

1. Context free languages provide a convenient notation for recursive description of languages.

2. The original goal of formalizing the structure of natural languages is still elusive, but CFGs are now the universally accepted formalism for definition of (the syntax of) *programming* languages.

3. Writing parsers has become an almost fully automated process thanks to this theory.

A Simple Grammar for English

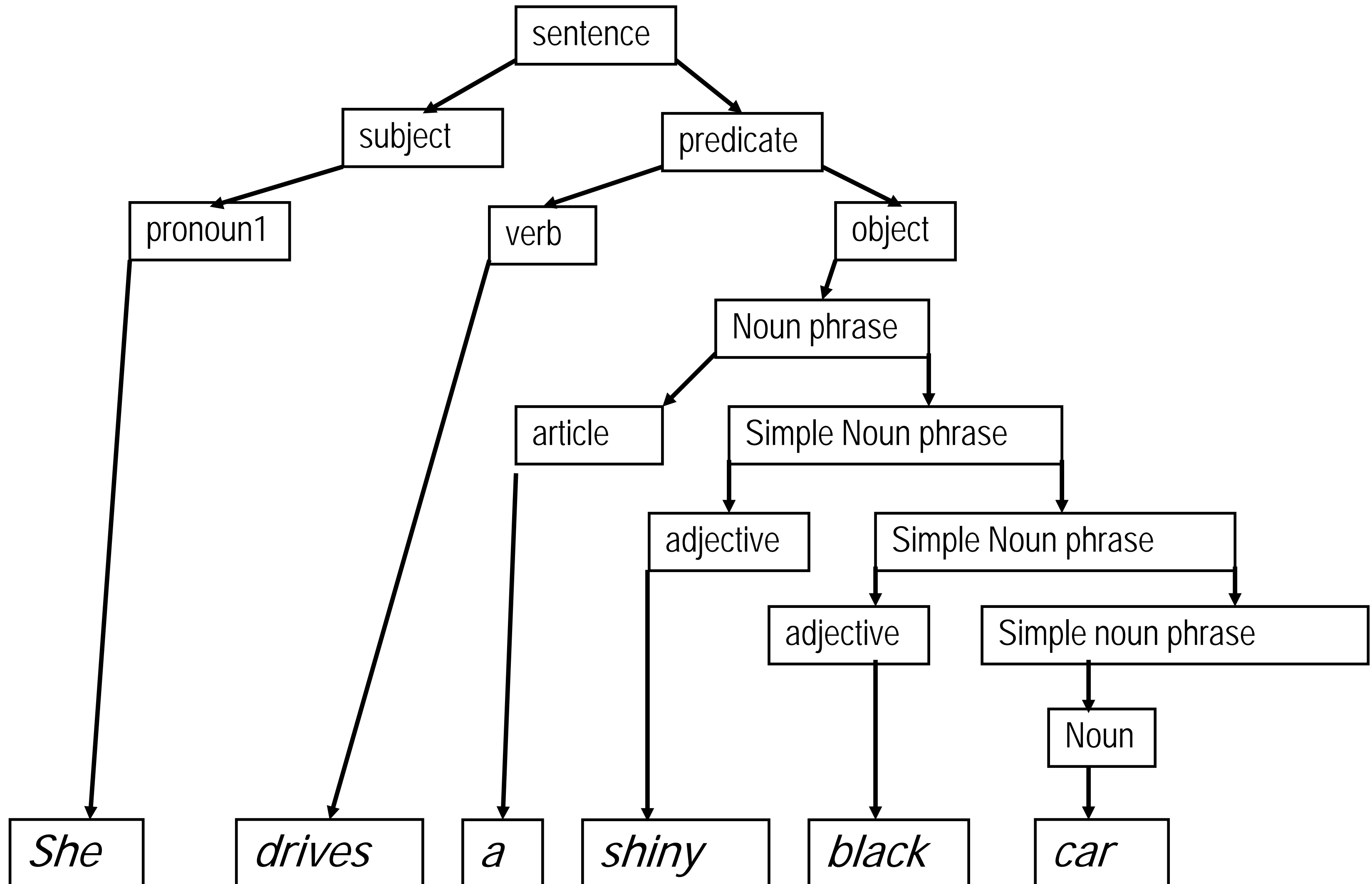
Example taken from Floyd & Beigel.

<Sentence>	→	<Subject> <Predicate>
<Subject>	→	<Pronoun1> | <Pronoun2>
<Pronoun1>	→	**I** | **we** | **you** | **he** | **she** | **it** | **they**
<Noun Phrase>	→	<Simple Noun Phrase> | <Article> <Noun Phrase>
<Article>	→	**a** | **an** | **the**
<Simple Noun Phrase>	→	<Noun> | <Adjective> <Simple Noun Phrase>
<Predicate>	→	<Verb> | <Verb> <Object>
<Object>	→	<Pronoun2> | <Noun Phrase>
<Pronoun2>	→	**me** | **us** | **you** | **him** | **her** | **it** | **them**
<Noun>	→	. . .
<Verb>	→	. . .

1. Each rule is called a production
2. lhs → lhs   lhs is a symbol, rhs is a string of symbols
3. Variable (or non-terminal) < … >
4. Symbol (or terminal) **bold**

Example

Derive the sentence "*She drives a shiny black car*" from these rules.



Derivation rules

1. Write down the start symbol (lhs of the first rule, unless otherwise stated)

2. Find a variable, v, and a corresponding rule where: $v \rightarrow rhs$

3. Replace the variable with the string rhs

4. Repeat, starting at 2. until no more variables remain.

Derivation

<sentence> ⇒

<subject> <predicate> ⇒

<pronoun> <predicate> ⇒

She <predicate> ⇒

She <verb> <object> ⇒

She drives <object> ⇒

She drives <simple noun phrase> ⇒

She drives <article> <noun phrase> ⇒

She drives a <noun phrase> ⇒

She drives a <adjective> <noun phrase> ⇒

She drives a shiny <noun phrase> ⇒

She drives a shiny <adjective> <simple noun phrase> ⇒

She drives a shiny black <simple noun phrase> ⇒

She drives a shiny black <noun> ⇒

She drives a shiny black car

I Have underlined the variable I will replace in the next step.

A Grammar for Expressions

<Expression>	→	<Term> | <Expression> **+** <Term>
<Term>	→	<Factor> | <Term> ***** <Factor>
<Factor>	→	<Identifer> | **(** <Expression> **)** | **3**
<Identifier>	→	**x** | **y** | **z** | …

In class exercise: Derive

- x + (y * 3)
- x + z * w + q

Definition of Context-Free-Grammars

A CFG is a quadruple G= (V,Σ,R,S), where

- V is a finite set of *variables* (*nonterminals*, *syntactic categories*)
- Σ is a finite set of *terminals*
- R is a finite set of *productions* -- rules of the form $X \longrightarrow a$, where $X \in V$ and $a \in (V \cup \Sigma)^*$
- S, the *start symbol*, is an element of V

Vertical bar (|), as used in the examples on the previous slide, is used to denote a set of several productions (with the same *lhs*).

Example

<Expression>	→	<Term> | <Expression> **+** <Term>
<Term>	→	<Factor> | <Term> ***** <Factor>
<Factor>	→	<Identifer> | **(** <Expression> **)** | **3**
<Identifier>	→	**x** | **y** | **z** | ...

V = {<Expression>, <Term>, <Factor>, <Identifier>}

Σ = {+, *, (,), x, y, z, 3,...}

R = {

<Expression> → <Term>
<Expression> → <Expression> **+** <Term>
<Term> → <Factor>
<Term> → <Term> ***** <Factor>
<Factor> → <Identifer>
<Factor> → **(** <Expression> **)**
<Identifier> → **x**
<Identifier> → **y**
<Identifier> → **z**
<Identifier> → ...

}

S = <Expression>

Notational Conventions

`a,b,c, …` (lower case, beginning of alphabet) are concrete terminals;

`u,v,w,x,y,z` (lower case, end of alphabet) are for strings of terminals

α, β, γ, … (Greek letters) are for strings over $(T \cup V)$ (*sentential forms*)

`A,B,C, …` (capitals, beginning of alphabet) are for variables (for non-terminals).

`X,Y,Z` are for variables standing for terminals.

Short-hand

Note. We often abbreviate a context free grammar, such as:

$G_2 = ($ $V=\{S\},$

$\Sigma=\{(,)\},$

$R=\{ S \to \varepsilon, S \to SS, S \to (S) \},$

$S=S\}$

By giving just its productions

$S \to \varepsilon \mid SS \mid (S)$

And by using the following conventions.

1) The start symbol is the lhs of the first production.
2) Multiple production for the same lhs non-terminal can be grouped together by using vertical bar (|)
3) Non-terminals are capitalized.
4) Terminal-symbols are lower case or non-alphabetic.

Definitions

The single-step derivation relation $\Rightarrow$ on $(V \cup T)^*$ is defined by:

1. $\alpha \Rightarrow \beta$ iff β is obtained from α by replacing an occurrence of the lhs of a production with its rhs. That is, $\alpha' A \alpha'' \Rightarrow \alpha' \gamma \alpha''$ is true iff $A \rightarrow \gamma$ is a production. We say $\alpha' A \alpha''$ yields $\alpha' \gamma \alpha''$

2. We write $\alpha \Rightarrow^* \beta$ when β can be obtained from α through a sequence of several (possibly zero) derivation steps.

3. The *language of the CFG* , G, is the set

$L(G) = \{w \in T^* \mid S \Rightarrow^* w\}$ (where S is the start symbol of G)

1. *Context-free languages* are languages of the form L(G)

Example 1

The familiar non-regular language

$$L = \{ a^k b^k \mid k \geq 0 \}$$

is context-free.

The grammar G_1 for it is given by $T=\{a,b\}$, $V=\{S\}$, and productions:

1. $S \rightarrow \varepsilon$
2. $S \rightarrow a\ S\ b$

Here is a derivation showing $a^3b^3 \in L(G)$:

$$S \Rightarrow_2 aSb \Rightarrow_2 aaSbb \Rightarrow_2 aaaSbbb \Rightarrow_1 aaabbb$$

(Note: we sometimes label the arrow with a subscript which tells the production used to enable the transformation)

Example 1 continued

Note, however, that the fact $L=L(G_1)$ is not totally obvious. We need to prove set inclusion both ways.

To prove $L \subseteq L(G_1)$ we must show that there exists a derivation for every string $a^k b^k$; this is done by induction on k.

For the converse, $L(G_1) \subseteq L$, we need to show that if $S \Rightarrow^* w$ and $w \in T^*$, then $w \in L$. This is done by induction on the length of derivation of w.

Example 2

The language of balanced parentheses is context-free. It is generated by the following grammar:

$$G_2 = (\; V=\{S\},$$
$$\Sigma=\{(,)\},$$
$$R=\{\; S \rightarrow \varepsilon \;\mid\; SS \;\mid (S)\},$$
$$S=S\}$$

Example 3

Consider the grammar:

$S \to AS \mid \varepsilon$

$A \to 0A1 \mid A1 \mid 01$

The derivation:

$S \Rightarrow AS \Rightarrow A1S \Rightarrow 011S \Rightarrow 011AS \Rightarrow 0110A1S \Rightarrow 0110011S \Rightarrow 0110011$

shows that $0110011 \in L(G_3)$.

Example 3 notes

The language $L(G_3)$ consists of strings $w \in \{0,1\}^*$ such that:

P(w): Either $w = \varepsilon$, or w begins with 0, and every block of 0's in w is followed by at least as many 1's

Again, the proof that G_3 generates all and only strings that satisfy P(w) is not obvious. It requires a two-part inductive proof.

Leftmost and Rightmost Derivations

The same string w usually has many possible derivations $S \equiv a_0 \Rightarrow a_1 \Rightarrow a_2 \Rightarrow \ldots \Rightarrow a_n \equiv w$

We call a derivation *leftmost* if in every step $a_i \Rightarrow a_{i+1}$, it is the first (leftmost) variable in $a_i\$$ that is being replaced with the rhs of a production. Similarly, in a *rightmost* derivation, it is always the last variable that gets replaced.

The above derivation of the string 0110011 in the grammar G_3 is leftmost. Here is a rightmost derivation of the same string:

$$S \Rightarrow A\underline{S} \Rightarrow AA\underline{S} \Rightarrow A\underline{A} \Rightarrow A0\underline{A}1 \Rightarrow \underline{A}0011 \Rightarrow \underline{A}10011 \Rightarrow 0110011$$

$S \rightarrow AS \mid \varepsilon$

$A \rightarrow 0A1 \mid A1 \mid 01$

Facts

Every Regular Language is also a Context Free Language

How might we prove this?

- Choose one of the many specifications for regular languages
- Show that every instance of that kind of specification has a total mapping into a Context Free Grammar

What is an appropriate choice?

Designing CFGs

Break the language into simpler (disjoint) parts with Grammars A B C . Then put them together $S \rightarrow Start_A \mid Start_B \mid Start_C$

If a Language fragment is Regular construct a DFA or RE, use these to guide you.

Infinite languages use rules like

$R \rightarrow a\ R$

$R \rightarrow R\ b$

Languages with linked parts use rules like

$R \rightarrow B\ x\ B$

$R \rightarrow x\ R\ x$

In Class Exercise

Map the Haskell Regular Expression datatype into a Context Free language.

```
data RegExp a
  = Lambda                          -- the empty string ""
  | Empty                           -- the empty set
  | One a                           -- a singleton set {a}
  | Union (RegExp a) (RegExp a)     -- union of two RegExp
  | Cat (RegExp a) (RegExp a)       -- Concatenation
  | Star (RegExp a)                 -- Kleene closure
```

Find CFG for these languages

$\{a^n\, b\, a^n \mid n \in Nat\}$

$\{ w \mid w \in \{a,b\}^*,$ and w is a palindrome of even length$\}$

$\{a^n\, b^k \mid n,k \in Nat, n \leq k\}$

$\{a^n\, b^k \mid n,k \in Nat, n \geq k\}$

$\{ w \mid w \in \{a,b\}^*,$ w has equal number of a's and b's $\}$

Ambiguity

A language is ambiguous if one of its strings has 2 or more leftmost (or rightmost) derivations.

- Consider the grammar
  1. E -> E + E
  2. E -> E * E
  3. E -> x | y
- And the string: x + x * y

$$E \Rightarrow_1 E + E \Rightarrow_3 x + E \Rightarrow_2 x + E * E \Rightarrow_3 x + x * E \Rightarrow_3 x + x * y$$

$$E \Rightarrow_2 E * E \Rightarrow_1 E + E * E \Rightarrow_3 x + E * E \Rightarrow_3 x + x * E \Rightarrow_3 x + x * y$$

Note we use the productions in a different order.

Common Grammars with ambiguity

Expression grammars with infix operators with different precedence levels.

Nested if-then-else statements

```
st  ->  if exp  then st  else st
        |  if exp then st
        | id := exp
```

if x=2 then if x=3 then y:=2 else y := 4

if x=2 then (if x=3 then y:=2) else y := 4
if x=2 then (if x=3 then y:=2 else y := 4)

Removing ambiguity.

Adding levels to a grammar

E -> E + E | E * E | id | (E)

Transform to an equivalent grammar

E -> E + T | T

T -> T * F | F

F -> id | (E)

Levels make formal the notion of precedence.
Operators that bind “tightly” are on the lowest levels

Chomsky Normal Form defined

There are many CFG's for any given CFL. When reasoning about CFL's, it often helps to assume that a grammar for it has some particularly simple form.

A grammar is in *Chomsky normal form* (CNF) if every rule is of the form

1. $A \rightarrow BC$ where B,C are variables
   1. B and C can't be S
2. $A \rightarrow a$ where a is a terminal
3. $S \rightarrow \varepsilon$ is allowed (only S can be nullable)

Remove ε-rules $A \rightarrow \varepsilon$

Remove unit rules $A \rightarrow B$

Transform rules with too many symbols on rhs

$A \rightarrow B\ B\ x\ C$

Finding a Chomsky Normal Form

Every grammar has a equivalent grammar (that generates the same language) in Chomsky normal form.

This grammar can be constructed using a few simple (language preserving) grammar transformations

ε-Productions

A variable A is *nullable* if $A \Rightarrow^* \varepsilon$. We can modify a given grammar G and obtain a grammar G' in which there are no nullable variables and which satisfies $L(G') = L(G) - \{\varepsilon\}$.

Find nullable symbols iteratively, using these facts:

1. If $A \rightarrow \varepsilon$ is a production, then A is nullable.
2. If $A \rightarrow B_1B_2 \ldots B_k$ is a production and $B_1, B_2, \ldots, B_k$ are all nullable, then A is nullable.

Once nullable symbols are known, we get G' as follows:

1. For every production $A \rightarrow \alpha$, add new productions $A \rightarrow \alpha'$, where α' is obtained by deleting some (or all) nullable symbols from α.
2. Remove all productions $A \rightarrow \varepsilon$

Example. If G contains a production $A \rightarrow BC$ and both B and C are nullable, then we add

$A \rightarrow B \mid C$

to G'.

Unit Productions

These are of the form $A \rightarrow B$, where A,B are variables.

Assuming the grammar has no ε–productions, we can eliminate unit productions as follows.

1. Find all pairs of variables such that $A \Rightarrow^* B$. (This happens iff B can be obtained from A by a chain of unit productions.)
2. Add new production $A \rightarrow \alpha$ whenever $A \Rightarrow^* B \Rightarrow \alpha$.
3. Remove all unit productions.

Theorem. For every CFG G, there exists a CFG G' in CNF such that L(G')=L(G) - {ε}

The first three steps of getting G' are elimination of ε-productions, elimination of unit productions, and elimination of useless symbols (in that order). There remain two steps:

1. Arrange that all productions are of the form $A \rightarrow \alpha$, where α is a terminal, or contains only variables.
2. Break up every production $A \rightarrow \alpha$ with $|\alpha|>2$ into productions whose rhs has length two.

For the first part, introduce a new variable C for each terminal c that occurs in the rhs of some production, add the production $C \to c$ (unless such a production already exists), and replace c with C in all other productions.

For example, the production $A \to 0B1$ would be replaced with $A_0 \to 0$, $A_1 \to 1$, $A \to A_0BA_1$.

An example explains the second part. The production $A \to BCDE$ is replaced by three others,

1. $A \to BA_1$,
2. $A_1 \to CA_2$,
3. $A_2 \to DE$,

using two new variables A_1, A_2.

Example page 110 Sipser

S -> a s a | Ab

A -> B | S

B -> b | ε

Rules

1. Add new start symbol
2. Remove ε
3. Remove unit rules
4. Make all rules have 1 terminal or 2 variables