Decidable and Undecidable Problems

Recall: Recognizable vs. Decidable

- A language L is Turing recognizable if some Turing machine recognizes it.
  - Some strings not in L may cause the TM to loop
  - Turing recognizable = **recursively enumerable (RE)**
- A language L is Turing decidable if some Turing machine decides it
  - To decide is to return a definitive answer; the TM must halt on all inputs
  - Turing decidable = decidable = **recursive**

Problems about Languages

- Consider some decision problems about languages, machines, and grammars:
  - Ex.: Is there an algorithm that given any DFA, M, and any string, w, tells whether M accepts w ?
  - Ex.: Is there an algorithm that given any two CFG's G_1 and G_2 tells whether $L(G_1) = L(G_2)$?
  - Ex. Is there an algorithm that given any TM, M, tells whether $L(M) = \emptyset$?
- By Church-Turing thesis: "is there an algorithm?" = "is there a TM?"

Machine encodings

- We can encode machine or grammar descriptions (and inputs) as strings over a **finite** alphabet.
  - Example: Let's encode the DFA $M = (Q,\Sigma,\delta,q_1,F)$ using the alphabet $\{0,1\}$
    - First, assign a unique integer ≥ 1 to each $q \in Q$ and $x \in \Sigma$
    - Code each transition $\delta(q_i,x_j) = q_k$ as $0^i 1 0^j 1 0^k$
    - Code $F = \{q_p,\ldots q_r\}$ as $0^p 1 \ldots 1 0^r$
    - Code M by concatenating codes for all transitions and F, separated by 11
  - We write $\langle M \rangle$ for the encoding of M and $\langle M,w \rangle$ for the encoding of M followed by input w

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Problems on encodings

- We can specify problems as languages over the encoding strings.
  - Ex.: $A_{DFA} = \{\langle M,w\rangle \mid M \text{ is a DFA that accepts } w\}$
  - Ex.: $EQ_{CFG} = \{\langle G,H\rangle \mid G \text{ and } H \text{ are CFG's and } L(G) = L(H)\}$
  - Ex.: $E_{TM} = \{\langle M\rangle \mid M \text{ is a TM and } L(M) = \emptyset\}$
- Now we can ask "is there a TM that **decides** this language?" (*i.e.*, is there an algorithm that solves this problem?)

A decidable language

- To show that a language is decidable, we have to describe an algorithm that decides it
  - We'll allow informal descriptions as long as we are confident they can in principle be turned into TMs
- Consider $A_{DFA} = \{ \langle M,w \rangle \mid M \text{ is a DFA that accepts } w \}$
- Algorithm: Check that M is a valid encoding; if not reject. Simulate behavior of M on w. If M halts in an accepting state, accept; if M halts in a rejecting state, reject.
  - We could easily write a C program that did this.

Another decidable language

- Consider $A_{CFG} = \{ \langle G,w \rangle \mid G \text{ is a CFG that generates } w \}$
- First attempt: build a TM that enumerates all possible derivations in G. If it finds w, it accepts. If it doesn't find w, it rejects.
- Problem: there may be an infinite number of derivations! So TM may never be able to reject.
- This TM **recognizes** A_{CFG}, but doesn't **decide** it.

Another try

- Consider $A_{ChCFG} = \{ \langle G,w\rangle \mid G$ is a CFG in Chomsky normal form that generates $w \}$
- We know that any derivation of w in G requires $2|w|-1$ steps (see the text, page 799) .
- So a TM that enumerates all derivations of this length can **decide** A_{ChCFG}.
- We also know an algorithm to convert an arbitrary CFG into CNF.
- Combining these two algorithms into a single TM gives a machine that **decides** A_{CFG}.

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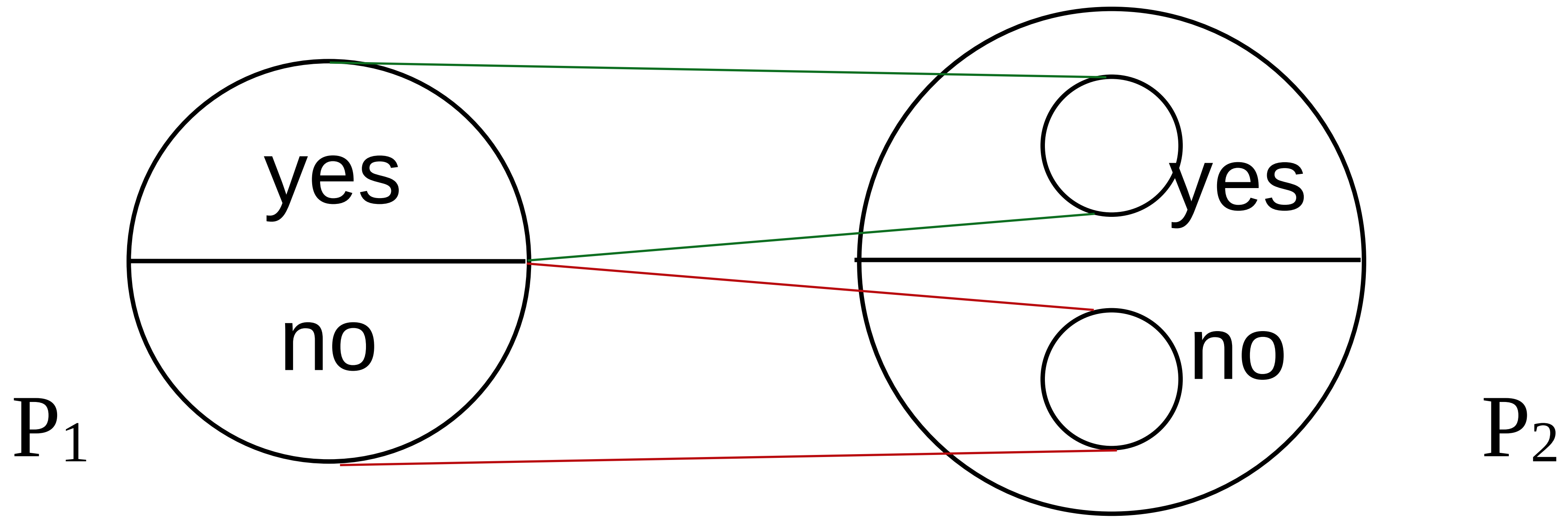
Reduction

- We solved the decision problem for A_{CFG} by algorithmically transforming the input into the form needed by another problem for which we could find a deciding TM.
- This strategy of **reducing** one problem P to another (known) problem Q is very common.
  - If P reduces to Q, and Q is decidable, then P is decidable.
- Must be certain that reduction process can be described by a TM !

Reductions (Hopcroft §9.3.1)



- Reductions must turn +ve instances of P_1 into +ve instances of P_2, -ve instances into -ve
- It's common that only a small part of P_2 be the target of the reduction.
- Reduction is a TM that translates an instance of P_1 into an instance of P_2

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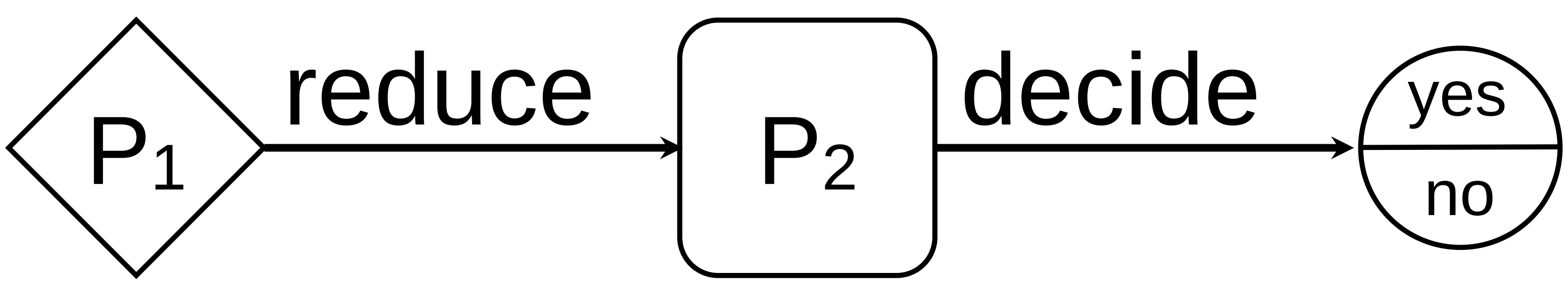
The Value of Reductions

If there is a reduction from P_1 to P_2, then:

1. If P_1 is undecidable, so is P_2
2. If P_1 is non-RE, then so is P_2



Proof by contradiction:

Suppose that P_2 is decidable …

then we can use P_2 to decide P_1

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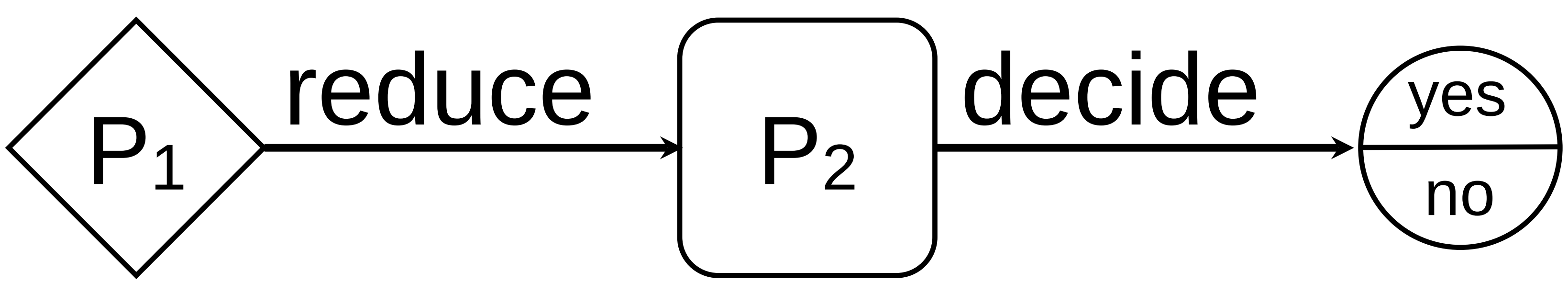

Proof by contradiction:

Suppose that P_2 is recognizable …

then we can use P_2 to recognize P_1

Some other decidable problems

- $A_{NFA} = \{\langle M,w\rangle \mid M \text{ is an NFA that accepts } w\}$
  - By direct simulation, or by reduction to A_{DFA}.
- $A_{REX} = \{\langle R,w\rangle \mid R \text{ is a regular expression that generates } w\}$
  - By reduction to A_{NFA}.
- $E_{DFA} = \{\langle M\rangle \mid M \text{ is a DFA and } L(D) = \emptyset\}$
  - By inspecting the DFA's transitions to see if there is any path to a final state.
- $EQ_{DFA} = \{\langle M_1,M_2\rangle \mid M_1, M_2 \text{ are DFA's and } L(M_1) = L(M_2)\}$
  - By reduction to E_{DFA}.
- $E_{CFG} = \{\langle G\rangle \mid G \text{ is a CFG and } L(G) = \emptyset\}$
  - By analysis of the CFG productions.

The Universal TM

- So far, we've fed descriptions of simple machines to TM's. But nothing stops us from feeding descriptions of TM's to TM's!
  - In fact, this is really what we've been leading up to
- A **universal** TM U behaves as follows:
  - U checks input has form ⟨M,w⟩ where M is an (encoded) TM and w is a string
  - U simulates behavior of M on input w.
  - If M ever enters an accept state, U accepts
  - If M ever rejects, U rejects

Role of Universal TM

- U models a (real-world) stored program computer.
  - Capable of doing many different tasks, depending on program you feed it
- Existence of U shows that the language $A_{TM} = \{\langle M,w \rangle \mid M \text{ is a TM and } M \text{ accepts } w\}$ is Turing-**recognizable**
- But it doesn't show that A_{TM} is Turing-**decidable**
  - If M runs forever on some w, U does too (rather than rejecting)

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A_{TM} is undecidable

- Proof is by contradiction.
- Suppose A_{TM} **is** decidable. Then some TM H decides it.
  - That is, for any TM M and input w, if we run H on ⟨M,w⟩ then H accepts if M accepts w and rejects if M does not accept w.
- Now use H to build a machine D, which
  - when started on input ⟨M⟩, runs H on ⟨M,⟨M⟩⟩
  - does the opposite of H: if H rejects, D accepts and if H accepts, D rejects.

H cannot exist

- We have

$$D(\langle M\rangle) = \begin{cases} accept & \text{if } M \text{ does not accept } \langle M\rangle \\ reject & \text{if } M \text{ accepts } \langle M\rangle. \end{cases}$$

- But now if we run D with its **own** description as input, we get

$$D(\langle D\rangle) = \begin{cases} accept & \text{if } D \text{ does not accept } \langle D\rangle \\ reject & \text{if } D \text{ accepts } \langle D\rangle. \end{cases}$$

- This is paradoxical! So D cannot exist. Therefore H cannot exist either. So A_{TM} is not decidable.

An unrecognizable language

- A language L is decidable $\Leftrightarrow$ both L and $\overline{L}$ are Turing-recognizable.
  - Proof: $\Rightarrow$ is obvious. For $\Leftarrow$, we have TM's M_1 and M_2 that recognize L, $\overline{L}$ respectively. Use them to build a TM M that runs M_1 and M_2 in parallel until one of them accepts (which must happen). If M_1 accepts M accepts too; if M_2 accepts, M rejects.
- $\overline{A_{TM}}$ is not Turing-recognizable.
  - Proof by contradiction. Suppose it is. Then, since A_{TM} is recognizable, A_{TM} is decidable. But it isn't!

$HALT_{TM}$ is undecidable

- $HALT_{TM} = \{\langle M,w\rangle \mid M$ is a TM and M halts on input $w\}$
- Proof is by **reduction** from A_{TM}.
- If problem P reduces to problem Q, and P is undecidable, then Q is undecidable!
  - Otherwise, we could use Q to decide P.
- So must show how a TM that decides $HALT_{TM}$ can be used to decide A_{TM}.

Acceptance reduces to Halting

- Assume TM R decides $HALT_{TM}$.
- Then the following TM S decides A_{TM}:
  - First, S runs R on ⟨M,w⟩.
  - If R rejects, we know that M does not halt on w. So M certainly does not accept w. So S rejects.
  - If R accepts, S simulates M on w until it halts (which it will!)
    - If M is in an accept state, S accepts; if M is in a reject state, S rejects.
- Since S cannot exist, neither can R.

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Another undecidable problem

- $E_{TM} = \{\langle M \rangle \mid M \text{ is a TM and } L(M) = \emptyset\}$ is undecidable.
- Proof is again by reduction from A_{TM}: we suppose TM R decides E_{TM} and use it to define a TM that decides A_{TM} as follows:
  - Check that input has form $\langle M,w \rangle$; if not, reject.
  - Construct a machine description $\langle M_1 \rangle$ such that $L(M_1) = L(M) \cap \{w\}$. (How?)
  - Run R on $\langle M_1 \rangle$. If it accepts, $L(M) \cap \{w\} = \emptyset$, so $w \notin L(M)$, so reject. If it rejects, $L(M) \cap \{w\} \neq \emptyset$, so $w \in L(M)$, so accept.

Rice's Theorem

- In fact, the approach of this last result can be generalized to prove **Rice's Theorem**:
- Let P be any **non-trivial** property of Turing-recognizable languages
  - Non-trivial means P is true of some but not all
- Then $\{\langle M \rangle \mid P \text{ is true of } L(M)\}$ is undecidable
- Examples of undecidable properties of L(M):
  - L(M) is empty, non-empty, finite, regular, CF, ...

Other Undecidable Problems

- Problems about CFGs G, G_1, G_2
  - Is G ambiguous?
  - Is $L(G_1) \subseteq L(G_2)$?
  - Is $\overline{L(G)}$ context-free?
- Post's Correspondence Problem
- Hilbert's 10th Problem
  - Does a polynomial equation $p(x_1, x_2, ..., x_n) = 0$ with integer coefficients have a solution consisting of integers?
- Equivalence Problem
  - Do two arbitrary Turing-computable functions have the same output on all arguments?

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Post's Correspondence Problem

- Given a finite sequence of pairs of strings (s_1, t_1), $(s_2, t_2), \ldots, (s_n, t_n)$, is there a sequence of indices $i_1, i_2, \ldots, i_k$ (duplications allowed) such that $s_{i1} s_{i2} \ldots s_{ik} = t_{i1} t_{i2} \ldots t_{ik}$?
- Example: (ab, a), (b, bb), (aa, b), (b, aab)
  - The sequence 1, 2, 1, 3, 4 gives us
    - abbabaab

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Post's Correspondence Problem

- Given a finite sequence of pairs of strings (s_1,t_1), (s_2,t_2),..., (s_n,t_n), is there a sequence of indices $i_1,i_2,...,i_k$ (duplications allowed) such that $s_{i1}s_{i2}...s_{ik} = t_{i1}t_{i2}...t_{ik}$?
- Example: (ab, a), (b, ab)
  - has no solution
  - Why?

Post's Correspondence Problem

- Given a finite sequence of pairs of strings (s_1, t_1), $(s_2, t_2), \ldots, (s_n, t_n)$, is there a sequence of indices $i_1, i_2, \ldots, i_k$ (duplications allowed) such that $s_{i1} s_{i2} \ldots s_{ik} = t_{i1} t_{i2} \ldots t_{ik}$?
- There is no algorithm that can decide, for an arbitrary instance of Post's Correspondence problem, whether there is a solution.

CS311 Computational Structures

The Halting Problem,

and other things uncomputable:

An approach by counting

Computability

- Anything computable can be computed by a Turing machine …
  - or one of the equivalent models, such as a partial recursive function or a λ-calculus expression
- But: not everything is computable
- Basic argument:
  - There are a countably-infinite number of Turing machines (partial recursive function, λ-calculus expressions…)
  - There are an uncountable number of functions $\mathbb{N} \rightarrow \mathbb{N}$

Countability of Turing Machines

- To prove that a set is countably infinite, we need only exhibit a bijection between its elements and $\mathbb{N}$
  - an *injection* suffices to show that it is countable
- That's called an "Effective Enumeration"
  - you have a way of "counting off" the Turing Machines
- Basic idea: you can encode anything (*e.g.*, a description of a Turing Machine) in binary
  - but any string of binary digits can be interpreted as a (large) integer

Hein's enumeration

- Take a (large) integer n
  - Write it in base-128 notation
  - regard each base-128 digit as an ASCII character
  - ask: is the resulting ascii string a description of a Turing machine?
- If so, that's the n^{th} Turing machine
- If not, arbitrarily say that the n^{th} Turing machine is "(0, a, a, S, Halt)"
- If we do this for all $n \in \mathbb{N}$, we will eventually get all the TMs

And for λ-calculus?

- All the expressions can also be effectively enumerated…
  - and also the primitive recursive functions,
  - and the Markov algorithms…
- The details are unimportant, so long as you agree that it makes sense to talk about the Turing machine (or λ-expression …) corresponding to a certain number.

Functions over the Natural Numbers

- There are an uncountable number of functions in $\mathbb{N} \rightarrow \mathbb{N}$
- We prove this by a diagnonalization argument
  - the same kind of argument that you used to prove that there were more real numbers than integers.
- Assume that there are a countable number of functions
- establish a contradiction
  - This is in chapter 2.4 of Hein (p.121) if you need to refresh your memory!

- Assume that $\mathbb{N} \to \mathbb{N}$ is countably infinite.
- Then there is a enumeration $f_0, f_1, f_2, f_3, \ldots$ of *all* of the functions in $\mathbb{N} \to \mathbb{N}$
- Now consider the function g: $\mathbb{N} \to \mathbb{N}$ defined as follows:
  - g n = if f_n n = 1 then 2 else 1
- Then g is not one of the f_i
  - it differs from f_0 at 0, from f_1 at 1, …
- This contradicts the assumption that $\mathbb{N} \to \mathbb{N}$ is countably infinite.

$\square$

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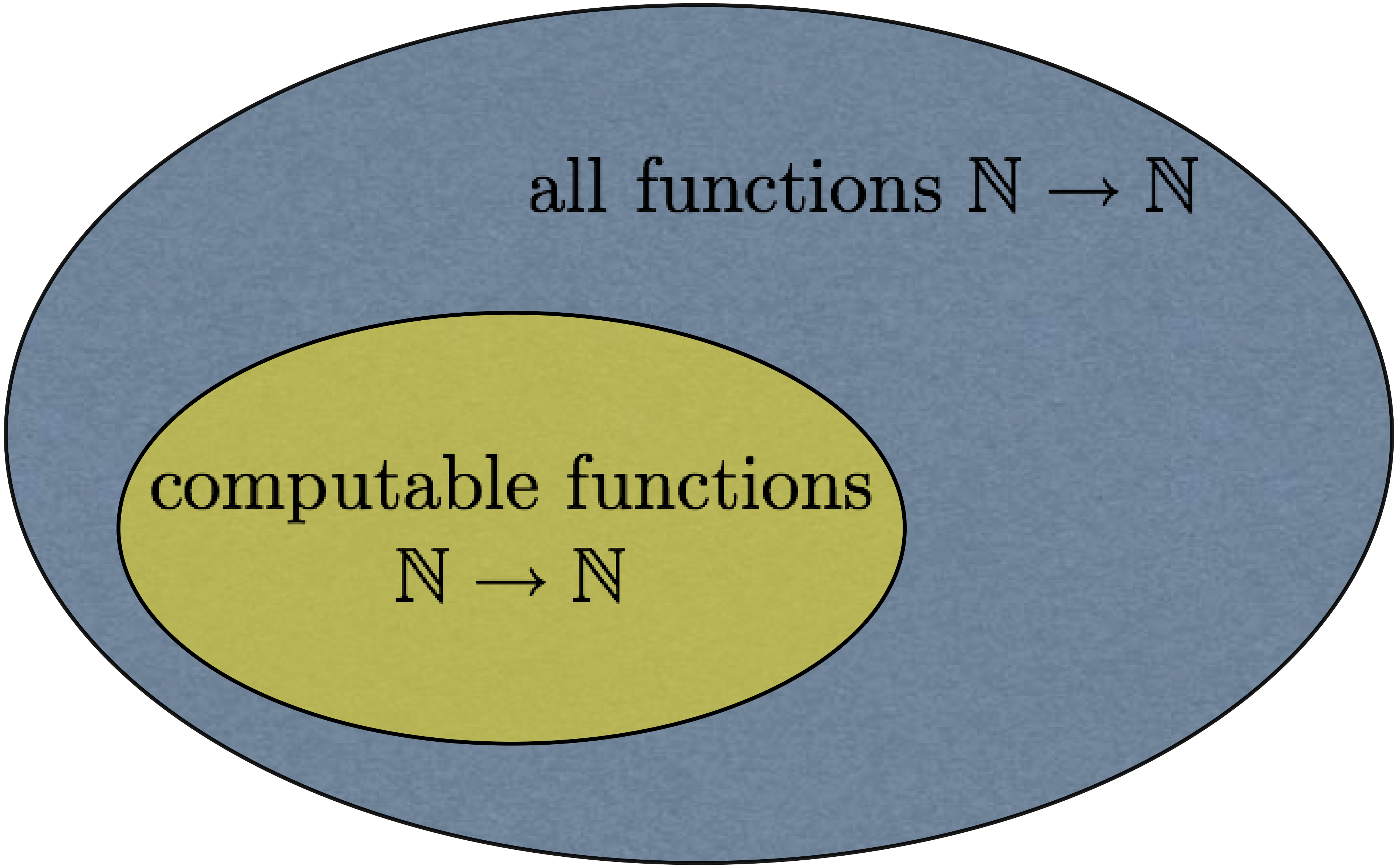
- There are *lots* of uncomputable functions
  - in fact: an uncountable number of them!



One Uncomputable Function

- Assume that the following function H(x) is computable
  - H(x) = if f_x halts on input X then loop forever else 0
- Then H must be in our enumeration of computable functions, say $H = f_k$
  - So: $f_k(x)$ = if f_x halts on input x then loop forever else 0
- Now apply f_k to its own index:
  - $f_k(k)$ = if f_k halts on input k then loop forever else 0
  - Thus: if $f_k(k)$ halts, then $f_k(k)$ loops forever, but if $f_k(k)$ loops forever, then $f_k(k) = 0$
- We have a contradiction

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The Halting Problem

- Is there a Turing Machine that can decide whether the execution of an arbitrary TM halts on an arbitrary input? No
- Is there a λ-calculus expression that can decide whether the application of an arbitrary λ-term to a second λ-term will reach a normal form? No
- Is there a simple program that can decide whether an arbitrary simple program will halt when given arbitrary initial values for its variables? No

What this *doesn't* mean

- Nothing about these results says that for *some* TM, or *some* simple program, or for *some* λ-expression, applied to *some* input, we can't decide whether it will halt.
- The unsolvability of the Halting problem just says that we can't *always* do it

Decidability

- A *decision problem* is a question with a *yes* or *no* answer
- The problem is *decidable* if there is an algorithm/function/TM that can input the problem and *always* halt with the correct answer
- The problem is semi-decidable (aka partially decidable, aka partially solvable) if there is an algorithm that halts and answers *yes* when the correct answer is yes, but may run forever if the answer is *no*.

Examples

- Is there an algorithm to decide if the following *simple* programs halt on arbitrary initial state:

```
X := 0
```

```
while X ≠ 0
do Y:= succ(X) od
```

- Is there an algorithm to decide if an arbitrary *simple* program halts on arbitrary initial state?
- What about Java programs? ML programs?

More Undecidable Problems

- Is there a Turing Machine that can recognize any Regular Language?
- Is there a Turing Machine that can recognize any Context-free language?
- Are all languages Turing-Recognizable?

What's a "Language"?

- A language over an alphabet A is a set of strings from A*
  - In other words: each subset of A* is a language
    - A language is a member of P(A*)
  - A* is countably infinite (for any finite A)
  - So P(A*) is uncountable
- There are an uncountable number of languages

Why is P(A*) Uncountable?

- The set B of infinite binary sequences is uncountable
- A* can be enumerated, say, in lexicographic order
- Any particular language, L, over A can be represented as a bit-mask, that is, as an element of B

Proof

- $A = \{a, b\}$
  $L_1 = \{ w \in A^* \mid w \text{ starts with a} \}$

- $A^* = \{ \Lambda, a, b, aa, ab, ba, bb, aaa, aab, \ldots \}$
  $L_1 = \{ \quad a, \quad aa, ab, \quad\quad aaa, aab, \ldots \}$
  $X(L_1) = \{ 0, 1, 0, 1, 1, 0, 0, 1, 1, \ldots \}$

✓ $X(L_1)$ is called the characteristic sequence of L_1

✓ $X(L_1)$ is a member of $\mathbb{B}$

✓ We have just displayed a bijection between $\mathbb{B}$ and the languages over A

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Some languages are not recognizable

- There are an uncountable number of languages
- There are a countable number of Turing machines
- Each Turing machine recognizes exactly one language

A Little History

- At the start of the 20th Century, it was thought that all mathematical problems were decidable
  - if you could formulate the problem precisely, and if you were smart enough, you could always come up with an algorithm to solve it.
- 1931: Kurt Gödel showed that this was *impossible* for arithmetic

Gödel’s Incompleteness Theorems

1. There are first-order statements about the natural numbers that can neither be proved nor disproved from Peano’s axioms
2. It’s impossible to prove from Peano’s axioms that Peano’s axioms are consistent.

- Two key ideas behind Gödel's proof

1. Gödel Numbering
   - Each formula (or sequence of formulae) can be encoded as an integer; each integer represents a formula or a sequence of formulae
   - So: $\omega(x, y)$ asserts that y is (the Gödel number of) a proof of x.
   - $\forall\, y \,.\, \neg\omega(x, y)$ asserts that x is unprovable.
2. Self reference (diagonalization)
   - if p is the Gödel number of $\forall\, y \,.\, \neg\omega(x, y)$, then $\zeta = \forall\, y \,.\, \neg\omega(p, y)$ asserts that ζ is unprovable

- This statement cannot be proved

- This statement is false
  XOR
  the earth is flat

Turing: applied Gödel to Computability

- The same two ideas:
  - Encoding: any “computing machine”, or program, can be represented as data (which the machine can take as input).
  - Self-reference: a machine (or program) operating on a description of itself as input

Halting Problem for Programmers

- Student claims that they have a program
  - fun halts(program, input): boolean = …
  - Note that halts takes an *encoding* of a program as its first argument.
- But look:
- paradox(program) = if halts(program, program)
  then loopForever
  else true
- paradox(paradox) answers *what* ?

- paradox(paradox) = if halts(paradox, paradox)
  then loopForever
  else true
  - If paradox halts when run on itself as input, then …
  - If paradox does not halt when run on itself as input …
- Either way, we have a contradiction
  - Therefore, you can't write the program halts